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EXAM FAM-L

Study Manual for SOA Exam FAM-L

1st Edition, 2nd Printing

by

Johnny Li, Ph.D., FSA
Andrew Ng, Ph.D., FSA

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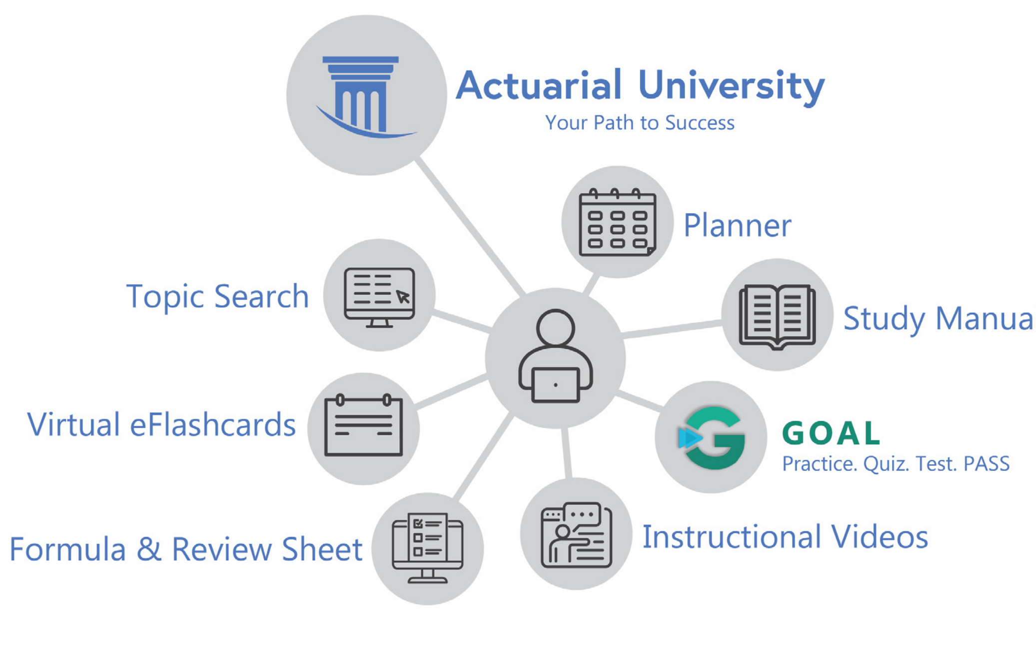

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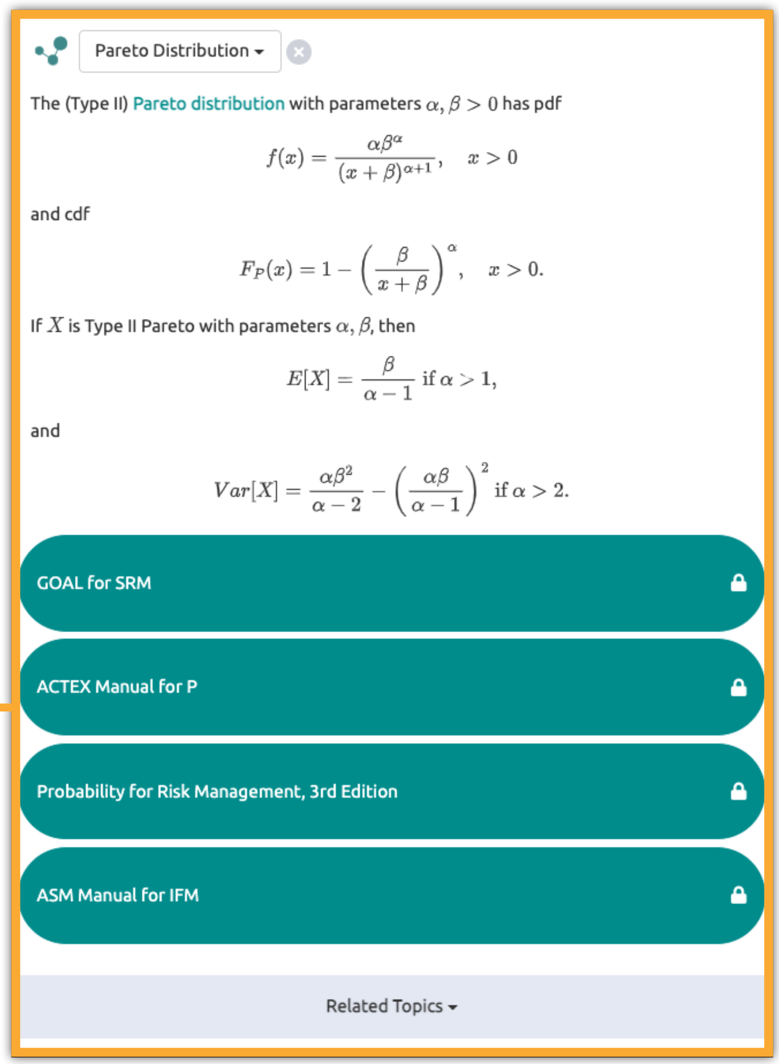

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Chapter 2

Life Tables

Objectives

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots,$ the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$ {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x}, $$

which means that we can calculate ${}_tp_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by ${}_td_x$. It immediately follows that ${}_td_x = l_x - l_{x+t}$.

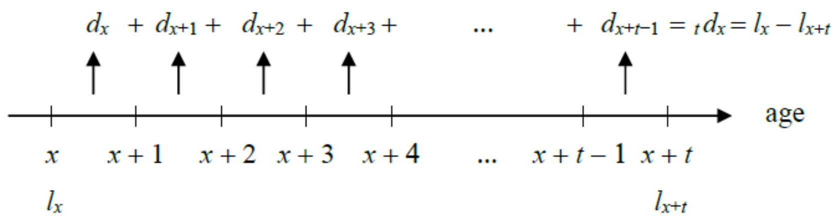
We can then calculate ${}_tq_x$ and ${}_{m|n}q_x$ by the following two relations:

$$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \quad {}_{m|n}q_x = \frac{{}_nd_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}. $$

When $t = 1$, we can omit the subscript t and write ${}_1d_x$ as d_x. By definition, we have

$$ {}_td_x = d_x + d_{x+1} + \cdots + d_{x+t-1}. $$

Graphically,



Also, when $t = 1$, we have the following relations:

$$ d_x = l_x - l_{x+1}, \ p_x = \frac{l_{x+1}}{l_x}, \text{ and } q_x = \frac{d_x}{l_x}. $$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities ${}_tp_x$ and death probabilities ${}_tq_x$ for all integral values of t and x easily.

Life Table Functions

(2.1) $$ {}_tp_x = \frac{l_{x+t}}{l_x} $$

(2.2) $$ {}_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \cdots + d_{x+t-1} $$

(2.3) $$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} $$

Some FAM-L exam and LTAM (the predecessor of FAM-L) questions are based on the Standard Ultimate Life Table, and some MLC (the predecessor of LTAM) exam questions are based on the Illustrative Life Table. The Standard Ultimate Life Table can be found at the SOA's website:

https://www.soa.org/Education/Exam-Req/edu-exam-ltam-detail.aspx,

and the Illustrative Life Table is provided in Appendix 2 of this study manual. The two tables have very similar formats. They contain a lot of information. For now, you only need to know and use the first three columns: x, l_x, and q_x (Standard Ultimate Life Table) and $1000q_x$ (Illustrative Life Table). For example, to obtain q_{43}, simply use the column labeled q_x. You should obtain $q_{43} = 0.000656$ (from the Standard Ultimate Life Table). It is also possible, but more tedious, to calculate q_{43} using the column labeled l_x; we have $q_{43} = 1 - l_{44}/l_{43} = 1 - 99104.3/99169.4 = 0.000656452$.

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have ${}_5p_{61} = l_{66}/l_{61} = 94020.3/96305.8 = 0.976268$ and ${}_5q_{61} = 1 - {}_5p_{61} = 1 - 0.976268 = 0.023732$ (from the Standard Ultimate Life Table). Here, you should not base your calculations on the column labeled q_x, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1.

You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) ${}_5p_{20}$

(b) q_{24}

(c) ${}_{4|1}q_{20}$

Solution:

(a) $${}_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{95650.15}{96178.01} = 0.994512.$$

(b) $$q_{24} = \frac{d_{24}}{l_{24}} = \frac{112.7102}{95762.86} = 0.001177.$$

(c) $${}_{4|1}q_{20} = \frac{{}_1d_{24}}{l_{20}} = \frac{112.7102}{96178.01} = 0.001172.$$

□

Example 2.2.

You are given:

(i) $S_0(x) = 1 - \frac{x}{100}, \quad 0 \le x \le 100$

(ii) $l_0 = 100$

(a) Find an expression for l_x for $0 \le x \le 100$.

(b) Calculate q_2.

(c) Calculate ${}_3q_2$.

Solution:

(a) $l_x = l_0 S_0(x) = 100 - x$.

(b) $q_2 = \frac{l_2 - l_3}{l_2} = \frac{98 - 97}{98} = \frac{1}{98}$.

(c) ${}_3q_2 = \frac{l_2 - l_5}{l_2} = \frac{98 - 95}{98} = \frac{3}{98}$.

□

In Exam FAM-L, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3.

For a certain population of 20 year olds, you are given:

(i) 2/3 of the population are nonsmokers, and 1/3 of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over [0, 80).

(iii) The future lifetime of a smoker is uniformly distributed over [0, 50).

Calculate ${}_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution: The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}\,l_{20}$ nonsmokers and $\frac{1}{3}\,l_{20}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 40 is $1-20/80 = 3/4$. For smokers, the proportion of individuals who can survive to age 40 is $1-20/50 = 3/5$. At age 40, there are $\frac{3}{4}\frac{2}{3}\,l_{20} = 0.5l_{20}$ nonsmokers and $\frac{3}{5}\frac{1}{3}\,l_{20} = 0.2l_{20}$ smokers. Hence, among those who can survive to age 40, 5/7 are nonsmokers and 2/7 are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over [0, 60). This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 - 5/60 = 11/12$.
- For a smoker at age 40, the remaining lifetime is uniformly distributed over [0, 30). This means that the probability for a smoker to survive from age 40 to age 45 is $1 - 5/30 = 5/6$.

Finally, for the whole population, we have

$$ {}_5p_{40} = \frac{5}{7} \times \frac{11}{12} + \frac{2}{7} \times \frac{5}{6} = \frac{25}{28}. $$

□

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of ${}_tp_x$ and ${}_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.

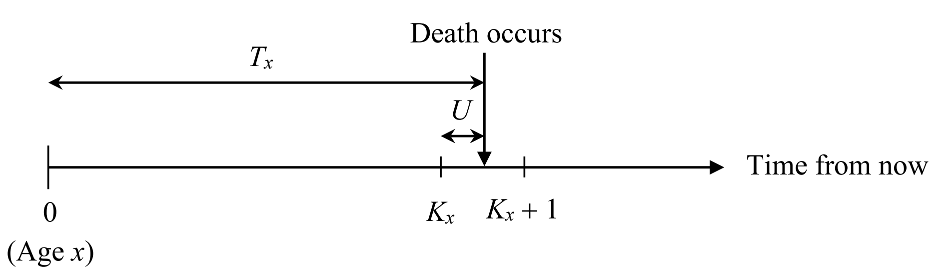
In Exam FAM-L, you are required to know two fractional age assumptions:

1. Uniform distribution of death
2. Constant force of mortality

We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam FAM-L syllabus. The idea behind this assumption is that we use a bridge, denoted by U, to connect the (continuous) future lifetime random variable T_x and the (discrete) curtate future lifetime random variable K_x. The idea is illustrated diagrammatically as follows:



It is assumed that U follows a uniform distribution over the interval $[0, 1]$, and that U and K_x are independent. Then, for $0 \le r < 1$ and an integral value of x, we have

$$\begin{aligned} {}_rq_x &= \Pr(T_x \le r) \\ &= \Pr(U < r \cap K_x = 0) \\ &= \Pr(U < r)\Pr(K_x = 0) \\ &= rq_x. \end{aligned}$$

The second last step follows from the assumption that U and K_x are independent, while the last step follows from the fact that U follows a uniform distribution over $[0, 1]$.

Key Equation for the UDD Assumption

(2.4) $${}_rq_x = rq_x, \quad \text{for } 0 \le r < 1$$

This means that under UDD, we have, for example, ${}_{0.4}q_{50} = 0.4q_{50}$. The value of q_{50} can be obtained straightforwardly from the life table. To calculate ${}_rp_x$, for $0 \le r < 1$, we use ${}_rp_x = 1 - {}_rq_x = 1 - rq_x$. For example, we have ${}_{0.1}p_{20} = 1 - 0.1q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between l_x and l_{x+1}, that is,

$$l_{x+r} = (1-r)l_x + rl_{x+1}.$$

Proof:

$$\begin{aligned} {}_rp_x &= 1 - rq_x = (1-r) + rp_x \\ \frac{l_{x+r}}{l_x} &= (1-r) + r\frac{l_{x+1}}{l_x} \\ l_{x+r} &= (1-r)l_x + rl_{x+1} \end{aligned}$$

□

You will find this equation – the interpolation between l_x and l_{x+1} – very useful if you are given a table of l_x (instead of q_x).

Application of the UDD Assumption to l_x

(2.5) $$l_{x+r} = (1-r)l_x + rl_{x+1}, \quad \text{for } 0 \le r < 1$$

What if the subscript on the left-hand-side of ${}_rq_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate ${}_{2.5}p_{30}$ as follows:

$${}_{2.5}p_{30} = {}_2p_{30} \times {}_{0.5}p_{32} = {}_2p_{30} \times (1 - 0.5q_{32}).$$

The value of ${}_2p_{30}$ and q_{32} can be obtained from the life table straightforwardly.

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider ${}_{0.1}p_{5.7}$ (both subscripts are not integers). The trick is that we multiply this probability with ${}_{0.7}p_5$, that is,

$${}_{0.7}p_5 \times {}_{0.1}p_{5.7} = {}_{0.8}p_5.$$

This gives ${}_{0.1}p_{5.7} = \dfrac{{}_{0.8}p_5}{{}_{0.7}p_5} = \dfrac{1-0.8q_5}{1-0.7q_5}$. The value of q_5 can be obtained from the life table.

To further illustrate this trick, let us consider ${}_{3.5}p_{4.6}$: This probability can be evaluated from the following equation:

$$ {}_{0.6}p_4 \times {}_{3.5}p_{4.6} = {}_{4.1}p_4. $$

Then, we have ${}_{3.5}p_{4.6} = \dfrac{{}_{4.1}p_4}{{}_{0.6}p_4} = \dfrac{{}_4p_4 \; {}_{0.1}p_8}{{}_{0.6}p_4} = \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$, and finally ${}_{3.5}q_{4.6} = 1-\dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$.

The values of ${}_4p_4$, q_8 and q_4 can be obtained from the life table.

Let us study the following example.

Example 2.4. You are given the following excerpt of a life table:

x	l_x	d_x
60	100000	300
61	99700	400
62	99300	500
63	98800	600
64	98200	700
65	97500	800

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) ${}_{0.26}p_{61}$

(b) ${}_{2.2}q_{60}$

(c) ${}_{0.3}q_{62.8}$

Solution:

(a) ${}_{0.26}p_{61} = 1 - {}_{0.26}q_{61} = 1 - 0.26 \times 400/99700 = 0.998957.$

Alternatively, we can calculate the answer by using a linear interpolation between l_{61} and l_{62} as follows:

$$ l_{61.26} = (1-0.26)l_{61} + 0.26 l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596. $$

It follows that ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99596/99700 = 0.998957.$

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_2p_{60} \times {}_{0.2}p_{62} = 1 - {}_2p_{60} \times (1 - 0.2q_{62})$

$$ = 1 - \frac{l_{62}}{l_{60}}\left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000}\left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008. $$

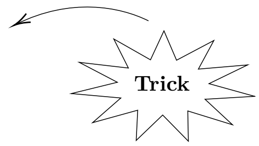
Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} as follows:

$$ l_{62.2} = (1-0.2)l_{62} + 0.2 l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200. $$

It follows that ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 1 - 99200/100000 = 0.008.$

(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{0.3}p_{62.8}$ from the following equation:

$$ {}_{0.8}p_{62} \times {}_{0.3}p_{62.8} = {}_{1.1}p_{62}. $$



Rearranging the equation above, we have

$$ {}_{0.3}p_{62.8} = \frac{{}_{1.1}p_{62}}{{}_{0.8}p_{62}} = \frac{p_{62}\ {}_{0.1}p_{63}}{{}_{0.8}p_{62}} = \frac{p_{62}\left(1-0.1q_{63}\right)}{1-0.8q_{62}} = \frac{\frac{98800}{99300}\left(1-0.1\times\frac{600}{98800}\right)}{1-0.8\times\frac{500}{99300}} $$

$$ = 0.998382. $$

Hence, ${}_{0.3}q_{62.8} = 1 - 0.998382 = 0.001618$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} and another interpolation between l_{63} and l_{64}:

First,

$$ l_{62.8} = (1-0.8)l_{62} + 0.8l_{63} = 0.2 \times 99300 + 0.8 \times 98800 = 98900. $$

Second,

$$ l_{63.1} = (1-0.1)l_{63} + 0.1l_{64} = 0.9 \times 98800 + 0.1 \times 98200 = 98740. $$

Finally,

$$ {}_{0.3}q_{62.8} = 1 - {}_{0.3}p_{62.8} = 1 - l_{63.1}/l_{62.8} = 1 - 98740/98900 = 0.001618. $$

□

Sometimes, you may be asked to calculate the density function of T_x and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

$$ f_x(r) = q_x, \quad 0 \le r < 1. $$

Proof: $f_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}F_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}\Pr(T_x \le r) = \frac{\mathrm{d}}{\mathrm{d}r}{}_rq_x = \frac{\mathrm{d}}{\mathrm{d}r}(rq_x) = q_x.$ □

Under UDD, we have the following equation for calculating the force of mortality:

$$ \mu_{x+r} = \frac{q_x}{1 - rq_x}, \quad 0 \le r < 1. $$

Proof: In general, $f_x(r) = {}_rp_x\mu_{x+r}$. Under UDD, we have $f_x(r) = q_x$ and ${}_rp_x = 1 - rq_x$. The result follows. □

Let us take a look at the following example.

Example 2.5. **[Course 3 Spring 2000 #12]**

For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$

(ii) $\mu_{81.5} = 0.0408$

(iii) $\mu_{82.5} = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Solution: The probability that a person age 80.5 will die within two years is ${}_2q_{80.5}$. We have

$${}_{0.5}p_{80} \times {}_2p_{80.5} = {}_{2.5}p_{80}.$$

This gives

$${}_2p_{80.5} = \frac{{}_2p_{80}\ {}_{0.5}p_{82}}{{}_{0.5}p_{80}} = \frac{p_{80}p_{81}\left(1-0.5q_{82}\right)}{1-0.5q_{80}} = \frac{\left(1-q_{80}\right)\left(1-q_{81}\right)\left(1-0.5q_{82}\right)}{1-0.5q_{80}}.$$

We then need to find q_{80}, q_{81} and q_{82} from the information given in the question. Using $\mu_{80.5}$, we have $\mu_{80.5} = \dfrac{q_{80}}{1-0.5q_{80}} \Rightarrow q_{80} = 0.0200$. Similarly, by using $\mu_{81.5}$ and $\mu_{82.5}$, we obtain $q_{81} = 0.0400$ and $q_{82} = 0.0600$.

Substituting q_{80}, q_{81} and q_{82}, we obtain ${}_2p_{80.5} = 0.921794$, and hence ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 0.0782$. Hence, the answer is (A).

□

Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age x, we approximate μ_{x+k} for $0 \le r < 1$ by a constant, which we denote by $\tilde{\mu}_x$. This means

$$\int_0^1 \mu_{x+u}\ \mathrm{d}u = \int_0^1 \tilde{\mu}_x\ \mathrm{d}u = \tilde{\mu}_x,$$

which implies $p_x = e^{-\tilde{\mu}_x}$ and $\tilde{\mu}_x = -\ln\left(p_x\right)$.

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued x, we have

$${}_rp_x = \left(p_x\right)^r, \quad 0 \le r < 1.$$

Proof: ${}_rp_x = \exp\left(-\int_0^r \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}u\right) = e^{-\tilde{\mu}_x r} = \left(e^{-\tilde{\mu}_x}\right)^r = \left(p_x\right)^r.$ □

For example, ${}_{0.3}p_{50} = \left(p_{50}\right)^{0.3}$, and ${}_{0.4}q_{62} = 1 - {}_{0.4}p_{62} = 1 - \left(p_{62}\right)^{0.4}$. We can generalize the equation above to obtain the following key formula.

Key Equation for the Constant Force of Mortality Assumption

(2.6) $\quad {}_r p_{x+u} = (p_x)^r, \quad \text{for } 0 \le r < 1 \text{ and } r+u \le 1$

Proof: ${}_r p_{x+u} = \exp\left(-\int_0^r \mu_{x+u+t}\mathrm{d}t\right) = \exp\left(-\int_0^r \tilde{\mu}_x \mathrm{d}t\right) = e^{-\tilde{\mu}_x r} = (p_x)^r.$

[The second step follows from the fact that given $0 \le r < 1$, $u+t$ is always less than or equal to 1 when $0 \le t \le r$.] □

Notice that the key equation for the constant force of mortality assumption is based on p, while that for the UDD assumption is based on q.

This key equation means that, for example, ${}_{0.2}p_{30.3} = (p_{30})^{0.2}$. Note that the subscript u on the right-hand-side does not appear in the result, provided that the condition $r+u \le 1$ is satisfied. But what if $r+u > 1$? The answer is very simple: Split the probability! To illustrate, let us consider ${}_{0.8}p_{30.3}$. (Here, $r+u = 0.8+0.3 = 1.1 > 1$.) By using equation (1.6) from Chapter 1, we can split ${}_{0.8}p_{30.3}$ into two parts as follows:

$${}_{0.8}p_{30.3} = {}_{0.7}p_{30.3} \times {}_{0.1}p_{31}.$$

We intentionally consider a duration of 0.7 years for the first part, because $0.3+0.7=1$, which means we can apply the key equation ${}_r p_{x+u} = (p_x)^r$ to it. As a result, we have

$${}_{0.8}p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.$$

The values of p_{30} and p_{31} can be obtained from the life table straightforwardly.

To further illustrate, let us consider ${}_{5.6}p_{40.8}$. We can split it as follows:

$${}_{5.6}p_{40.8} = {}_{0.2}p_{40.8} \times {}_{5.4}p_{41} = {}_{0.2}p_{40.8} \times {}_5p_{41} \times {}_{0.4}p_{46} = (p_{40})^{0.2} \times {}_5p_{41} \times (p_{46})^{0.4}.$$

The values of p_{40}, ${}_5p_{41}$ and p_{46} can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for $0 \le r < 1$, the value of $\ln(l_{x+r})$ can be obtained by a linear interpolation between the values of $\ln(l_x)$ and $\ln(l_{x+1})$.

Proof: Setting $u = 0$ in equation (2.6), we have

$$\begin{aligned} {}_r p_x &= (p_x)^r \\ \frac{l_{x+r}}{l_x} &= \left(\frac{l_{x+1}}{l_x}\right)^r \\ \ln(l_{x+r}) - \ln(l_x) &= r\ln(l_{x+1}) - r\ln(l_x) \\ \ln(l_{x+r}) &= (1-r)\ln(l_x) + r\ln(l_{x+1}) \end{aligned}$$

□

You will find this equation – the interpolation between $\ln(l_x)$ and $\ln(l_{x+1})$ – useful when you are given a table of l_x.

Application of the Constant Force of Mortality Assumption to l_x

$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1}), \qquad \text{for } 0 \le r < 1$$

Example 2.6.

Assuming constant force of mortality between integral ages, repeat Example 2.4.

Solution:

(a) ${}_{0.26}p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955.$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1-0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99595.84526/99700 = 0.998955$.

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_{2}p_{60} \times {}_{0.2}p_{62} = 1 - {}_{2}p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}}\left(\frac{l_{63}}{l_{62}}\right)^{0.2} = 1 - \frac{99300}{100000}\left(\frac{98800}{99300}\right)^{0.2} = 0.008002.$$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{62})$ and $\ln(l_{63})$ as follows: $\ln(l_{62.2}) = (1-0.2)\ln(l_{62}) + 0.2\ln(l_{63})$, which gives $l_{62.2} = 99199.79798$. Hence, ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 0.008002$.

(c) First, we consider ${}_{0.3}p_{62.8}$:

$${}_{0.3}p_{62.8} = {}_{0.2}p_{62.8} \times {}_{0.1}p_{63} = (p_{62})^{0.2}(p_{63})^{0.1}.$$

Hence,

$${}_{0.3}q_{62.8} = 1 - (p_{62})^{0.2}(p_{63})^{0.1} = 1 - \left(\frac{l_{63}}{l_{62}}\right)^{0.2}\left(\frac{l_{64}}{l_{63}}\right)^{0.1}$$

$$= 1 - \left(\frac{98800}{99300}\right)^{0.2}\left(\frac{98200}{98800}\right)^{0.1} = 0.001617.$$

Alternatively, we can calculate the answer by an interpolation between $\ln(l_{62})$ and $\ln(l_{63})$ and another interpolation between $\ln(l_{63})$ and $\ln(l_{64})$.

First, $\ln(l_{62.8}) = (1-0.8)\ln(l_{62}) + 0.8\ln(l_{63})$, which gives $l_{62.8} = 98899.79818$.

Second, $\ln(l_{63.1}) = (1-0.1)\ln(l_{63}) + 0.1\ln(l_{64})$, which gives $l_{63.1} = 98739.8354$.

Finally, ${}_{0.3}q_{62.8} = 1 - l_{63.1}/l_{62.8} = 0.001617$.

□

Example 2.7.

You are given the following life table:

x	l_x	d_x
90	1000	50
91	950	50
92	900	60
93	840	c_1
94	c_2	70
95	700	80

(a) Find the values of c_1 and c_2.

(b) Calculate ${}_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.

Solution:

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

$$\begin{aligned} {}_{1.4}p_{90} &= p_{90} \times {}_{0.4}p_{91} \\ &= p_{90}\,(1 - 0.4q_{91}) \\ &= \frac{l_{91}}{l_{90}}\left(1 - 0.4\frac{d_{91}}{l_{91}}\right) \\ &= \frac{950}{1000}\left(1 - 0.4 \times \frac{50}{950}\right) \\ &= 0.93. \end{aligned}$$

Alternatively, you can compute the answer by interpolating between l_{92} and l_{91}:

$${}_{1.4}p_{90} = p_{90} \times {}_{0.4}p_{91} = \frac{l_{91}}{l_{90}}\left(\frac{0.4l_{92} + 0.6l_{91}}{l_{91}}\right) = \frac{0.4 \times 900 + 0.6 \times 950}{1000} = 0.93$$

(c) Assuming constant force of mortality between integer ages, we have

$$\begin{aligned} {}_{1.4}p_{90} &= p_{90} \times_{0.4} p_{91} \\ &= p_{90} \times (p_{91})^{0.4} \\ &= \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} \\ &= 0.92968. \end{aligned}$$

□

Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

	UDD	**Constant force**
${}_rp_x$	$1 - rq_x$	$(p_x)^r$
${}_rq_x$	rq_x	$1 - (p_x)^r$
μ_{x+r}	$\dfrac{q_x}{1 - rq_x}$	$-\ln(p_x)$

In the table, x is an integer and $0 \le r < 1$. The shaded formulas are the key formulas that you must remember for the examination.

2.3 Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called "guaranteed issue" which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.

- $[x]$ indicates the age at selection (i.e., the age at which the underwriting was done).
- $[x]+t$ indicates a person currently age $x+t$ who was selected at age x (i.e., the underwriting was done at age x). This implies that the insurance contract has elapsed for t years.

For example, we have the following select probabilities:

- $q_{[x]}$ is the probability that a life age x now dies before age $x+1$, given that the life is selected at age x.
- $q_{[x]+t}$ is the probability that a life age $x+t$ now dies before age $x+t+1$, given that the life was selected at age x.

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability q_{x+t}. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is n years, we have

$$q_{[x]+t} < q_{x+t}, \quad \text{for } t < n.$$
$$q_{[x]+t} = q_{x+t}, \quad \text{for } t \geq n.$$

The ordinary death probability q_{x+t} is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
40	0.04	0.06	0.08	42
41	0.05	0.07	0.09	43
42	0.06	0.08	0.10	44
43	0.07	0.09	0.11	45

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: $q_{[41]} = 0.05$

Age 42: $q_{[41]+1} = 0.07$

Age 43: $q_{[41]+2} = q_{43} = 0.09$

Age 44: $q_{[41]+3} = q_{44} = 0.10$

Age 45: $q_{[41]+4} = q_{45} = 0.11$

As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{[40]+1} = 0.06$

Age 42: $q_{[40]+2} = q_{42} = 0.08$

Age 43: $q_{[40]+3} = q_{43} = 0.09$

Age 44: $q_{[40]+4} = q_{44} = 0.10$

Age 45: $q_{[40]+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x,k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}.$$

For example, on the basis of the preceding table, $I(41,1) = 1 - q_{[41]+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to q_{x+k}, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to q_{x+k}, and therefore $I(x,k)$ would be close to zero.

Let us first go through the following example, which involves a table of $q_{[x]}$.

Example 2.8. **[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.

(iii) White's age on 01/01/2001 is 61.

(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \leq P < 0.43$

(B) $0.43 \leq P < 0.45$

(C) $0.45 \leq P < 0.47$

(D) $0.47 \leq P < 0.49$

(E) $0.49 \leq P < 1.00$

Solution: White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = {}_5p_{[60]+1}$. We have

$$\begin{aligned}
P &= {}_5p_{[60]+1} \\
&= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5} \\
&= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65} \\
&= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65}) \\
&= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17) \\
&= 0.4589.
\end{aligned}$$

Hence, the answer is (C).

□

In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

Example 2.9. **[MLC Spring 2012 #13]**

Lorie's Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

Limousine age (x)	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
0	0.100	0.167	0.333	2
1	0.100	0.333	0.500	3
2	0.150	0.400	1.000	4
3	0.250	0.750	1.000	5
4	0.500	1.000	1.000	6
5	1.000	1.000	1.000	7

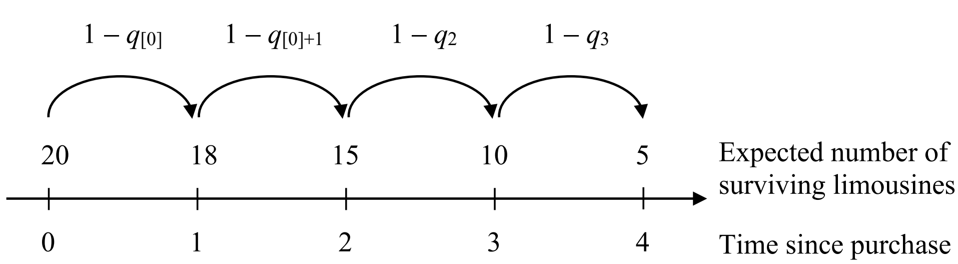
Lorie's Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie's Lorries fleet immediately after the purchase of this year's limousines.

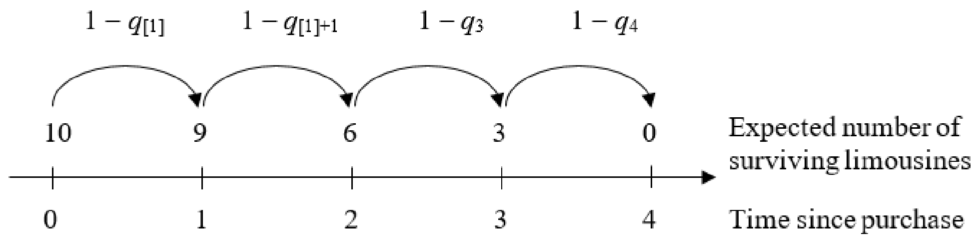
(A) 93 (B) 94 (C) 95 (D) 96 (E) 97

Solution: Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

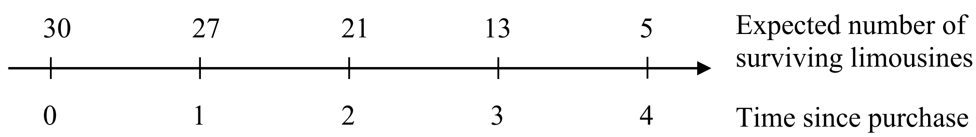
For the 20 brand new limousines, their "age at selection" is 0. As such, the sequence of "death" probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, q_2, q_3, q_4, q_5, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most four years since the time of purchase. For these 20 brand new limousines, the expected number of "survivors" limousines in each future year can be calculated as follows:



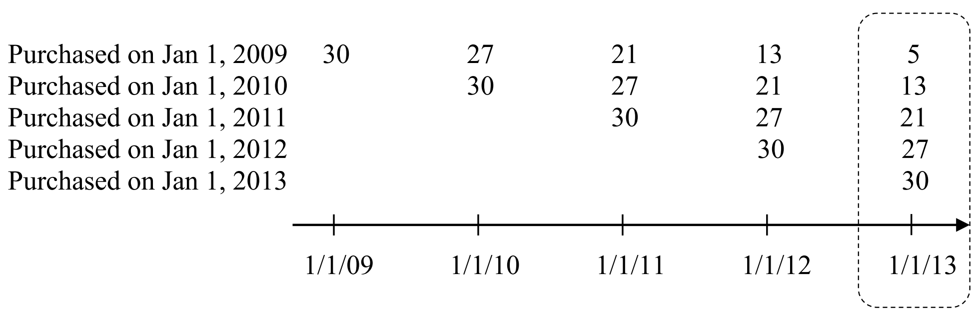
For the 10 1-year-old limousines, their "age at selection" is 1. As such, the sequence of "death" probabilities applicable to these 10 1-year-old limousines are $q_{[1]}$, $q_{[1]+1}$, q_3, q_4, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of "surviving" limousines in each future year can be calculated as follows:



Considering the entire purchase of 30 limousines, we have the following:



Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie's fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:



The answer is thus $5 + 13 + 21 + 27 + 30 = 96$, which corresponds to option (D).

□

Sometimes, you may be given a select-and-ultimate table that contains the life table function l_x. In this case, you can calculate survival and death probabilities by using the following equations:

$$_sp_{[x]+t} = \frac{l_{[x]+t+s}}{l_{[x]+t}}, \quad _sq_{[x]+t} = 1 - \frac{l_{[x]+t+s}}{l_{[x]+t}}.$$

Let us study the following two examples.

Example 2.10. You are given the following select-and-ultimate table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9900	9897	33
32	9899	9896	9892	34
33	9894	9891	9887	35

Calculate the following:

(a) ${}_2q_{[31]}$

(b) ${}_2p_{[30]+1}$

(c) ${}_{1|2}q_{[31]+1}$

Solution:

(a) $${}_2q_{[31]} = 1 - \frac{l_{[31]+2}}{l_{[31]}} = 1 - \frac{l_{33}}{l_{[31]}} = 1 - \frac{9897}{9903} = 0.000606.$$

(b) $${}_2p_{[30]+1} = \frac{l_{[30]+1+2}}{l_{[30]+1}} = \frac{l_{33}}{l_{[30]+1}} = \frac{9897}{9905} = 0.999192.$$

(c) $${}_{1|2}q_{[31]+1} = \frac{l_{[31]+1+1} - l_{[31]+1+1+2}}{l_{[31]+1}} = \frac{l_{33} - l_{35}}{l_{[31]+1}} = \frac{9897 - 9887}{9900} = 0.001010.$$ □

Exam questions such as the following may involve both $q_{[x]}$ and $l_{[x]}$.

Example 2.11. **[MLC Spring 2012 #1]**

For a 2-year select and ultimate mortality model, you are given:

(i) $q_{[x]+1} = 0.95q_{x+1}$

(ii) $l_{76} = 98{,}153$

(iii) $l_{77} = 96{,}124$

Calculate $l_{[75]+1}$.

(A) 96,150 (B) 96,780 (C) 97,420 (D) 98,050 (E) 98,690

Solution: From (ii) and (iii), we know that $q_{76} = 1 - 96124/98153 = 0.020672$.

From (i), we know that $q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638$.

Since

$$l_{[75]+2} = l_{[75]+1}(1 - q_{[75]+1}),$$

and $l_{[75]+2} = l_{77}$, we have $l_{[75]+1} = 96124/(1 - 0.019638) = 98049.5$. The answer is (D). □

Mock Test 1

****BEGINNING OF EXAMINATION****

1. For a whole life insurance of 1,000 issued to life selected at age x,

   (i) Percent of premium expenses is 90% in the first year, and 10% in each year thereafter.

   (ii) Maintenance expenses are 15 per 1,000 of insurance in the first year, and 3 per 1,000 of insurance thereafter.

   (iii) Claim settlement expenses are 10 per 1,000 of insurance.

   (iv) A 15-year select and ultimate mortality is to be used.

   Determine the expression of the gross premium for the policy using equivalence principle.

   (A) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (B) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

   (C) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{\ddot{a}_{[x]} - 0.9}$

   (D) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (E) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

2. The following results were obtained from a survival study, using the Kaplan-Meier (KM) estimator:

Time of death t in the sample	KM estimate of $S(t)$	Standard error of estimate
17	0.957	0.0149
25	0.888	0.0236
32	0.814	0.0298
36	0.777	0.0321
39	0.729	0.0348
42	0.680	0.0370
44	0.659	0.0378
47	0.558	0.0418
50	0.360	0.0470
54	0.293	0.0456
56	0.244	0.0440
57	0.187	0.0420
59	0.156	0.0404
62	0.052	0.0444

   Find the upper limit of the 95% log-confidence interval for $S(45)$.

   (A) 0.70 (B) 0.71 (C) 0.72 (D) 0.73 (E) 0.74

3. You are given:

(i) $i = 0.07$

(ii) $\ddot{a}_x = 11.7089$

(iii) $\ddot{a}_{x:\overline{40}|} = 11.55$

(iv) $\ddot{a}_{x+40} = 7.1889$

Find $A^{1}_{x:\overline{40}|}$.

(A) 0.120 (B) 0.147 (C) 0.172 (D) 0.197 (E) 0.222

4. You are given the following select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.11	0.13	0.15	67
66	0.12	0.135	0.16	68
67	0.13	0.145	0.17	69

Deaths are uniformly distributed over each year of age.

Find ${}_{1.8}p_{[66]+0.6}$.

(A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

5. For a 3-year fully discrete endowment insurance of 1,000 on (x), you are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.15$

(iii) $v = 0.9$

(iv) Deaths are uniformly distributed over each year of age.

Calculate the net premium policy value 9 months after the issuance of the policy.

(A) 274 (B) 280 (C) 283 (D) 286 (E) 290

6. Which of the following statements is/are correct?

I. Insurable interest in an entity exists if one would suffer a financial loss if that entity is damaged.

II. Insurable interest is related to the concept of adverse selection.

III. Stranger owned life insurance is illegal in many jurisdiction because the purchaser has no insurable interest in the insured.

(A) I only (B) II only (C) I and III only

(D) II and III only (E) I, II and III

7. Which of the following is/are strictly increasing function(s) of T_x for all $T_x \geq 0$?

I. The present value random variable for a continuous whole life annuity of \$1 on (x)

II. The present value random variable for a continuous n-year temporary life annuity of \$1 on (x)

III. The net future loss at issue random variable for a fully continuous whole life insurance of \$1 on (x)

(A) I only (B) III only (C) I, II only
(D) I, III only (E) I, II and III

8. For a fully discrete whole life insurance on (30), you are given:

(i) $i = 0.05$

(ii) $q_{29+h} = 0.004$

(iii) The net amount at risk for policy year h is 1295.

(iv) The terminal policy value for policy year $h-1$ is 179.

(v) $\ddot{a}_{30} = 16.2$

Calculate the initial policy value for policy year $h+1$.

(A) 188 (B) 192 (C) 200 (D) 214 (E) 226

9. You are given:

(i) $\mu_{x+t} = \begin{cases} 0.02 & 0 \leq t < 1 \\ 0.07 & 1 \leq t < 2 \end{cases}$ (ii) $Y = \min(T_x, 2)$

Calculate E(Y).

(A) 1.88 (B) 1.90 (C) 1.92 (D) 1.94 (E) 1.96

10. An insurer issues fully discrete endowment insurance policies to a group of 50 high-risk drivers all aged 35 with a sum insured of 2,000. You are given:

(i) The mortality of each driver follows the Standard Ultimate Life Table with an age rating of 3 years. That is,

$$q_x = q_{x+3}^{SULT},$$

where q_y^{SULT} is the 1-year death probability under the Standard Ultimate Life Table.

(ii) Lifetimes of the group of 50 high-risk drivers are independent.

(iii) Premiums are payable annually in advance. Each premium is 110% of the net annual premium.

(iv) $i = 0.05$

By assuming a normal approximation, estimate the 95$^{\text{th}}$-percentile of the net future loss random variable.

(A) 1090 (B) 1240 (C) 1390 (D) 1540 (E) 1690

11. For a population, the force of mortality is $\mu_x = 0.01\pi \tan(0.01\pi x)$ for $0 < x < 50$.

Calculate ${}_{10|5}q_{20}$.

[Note: The anti-derivative of $\tan x$ is $-\ln|\cos x|$.]

(A) 0.15 (B) 0.17 (C) 0.19 (D) 0.21 (E) 0.23

12. Which of the following regarding is/are correct?

I. Variable annuity has cash value.

II. Term insurance has cash value.

III. Universal life insurance has cash value.

(A) I only (B) II only (C) III only

(D) II and III only (E) I, II and III

13. You are given the following about 100 insurance policies in a study of time to policy surrender:

(i) The study was designed in such a way that for every policy that was surrendered, a new policy was added, meaning that the risk set, r_j, is always equal to 100.

(ii) Policies are surrendered only at the end of a policy year.

(iii) The number of policies surrendered at the end of each policy year was observed to be:

2 at the end of the 1^{st} policy year

3 at the end of the 2^{nd} policy year

4 at the end of the 3^{rd} policy year

$\vdots$

$n+1$ at the end of the n^{th} policy year

(iv) The Nelson-Aalen empirical estimate of the cumulative distribution function at time n is 0.41725.

Find n.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

14. You are given:

(i) $\delta = 0.05$

(ii) $\bar{A}_x = 0.44$

(iii) ${}^2\bar{A}_x = 0.22$

Consider a portfolio of 100 fully continuous whole life insurances. The ages of the all insureds are x, and their lifetimes are independent. The face amount of the policies, the premium rate and the number of policies are as follows:

Face amount	Premium rate	Number of Policies
100	4.3	75
400	17.5	25

By using a normal approximation, calculate the probability that the present value of the aggregate loss-at-issue for the insurer's portfolio will exceed 700.

(A) $1 - \Phi(2.28)$ (B) $1 - \Phi(0.17)$ (C) $\Phi(0)$
(D) $\Phi(0.17)$ (E) $\Phi(2.28)$

15. Victor is now age 22, and his future lifetime has the following cumulative distribution function:

$$\mathrm{F}_{22}(t) = 1 - (1 + 0.04t)e^{-0.04t}.$$

Let Z be the present value random variable for a fully continuous life insurance that pays 100 immediately on the death of Victor provided that he dies between ages 32 and 52.

The force of interest is 0.06.

Find the 70^{th}-percentile of Z.

(A) 0 (B) 25 (C) 40 (D) 47 (E) 50

16. You are given:

(i) $i = 0.05$

(ii) $A_{60} = 0.560$

(iii) $A_{40} = 0.176$

(iv) ${}_{20}p_{40} = 0.75$

Calculate $a^{(3)}_{40:\overline{20}|}$ using the two-term Woolhouse's approximation.

(A) 13.7 (B) 14.2 (C) 14.5 (D) 14.9 (E) 15.1

17. You are given:

(i) $\delta = 0.05$

(ii) $\ddot{a}_{60} = 12.18$

(iii) $p_{60} = 0.98$

Using the claims accelerated approach, calculate $\ddot{a}_{61}^{(6)}$.

(A) 11.6 (B) 11.7 (C) 11.8 (D) 11.9 (E) 12.1

18. Let Y be the present value random variable for a special three-year temporary life annuity on (x). You are given:

(i) The life annuity pays $2 + k$ at time k, for $k = 0$, 1 and 2.

(ii) $v = 0.9$

(iii) $p_x = 0.8$, $p_{x+1} = 0.75$, $p_{x+2} = 0.5$

Calculate the standard deviation of Y.

(A) 1.2 (B) 1.8 (C) 2.4 (D) 3.0 (E) 3.6

19. You are given:

$$\mu_x = \begin{cases} 0.04 & 50 \leq x < 60 \\ 0.05 + 0.001(x - 60)^2 & 60 \leq x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38 (B) 0.44 (C) 0.47 (D) 0.50 (E) 0.56

20. You are given:

(i) $p_{41} = 0.999422$

(ii) $i = 0.03$

(iii) $\ddot{a}_{42:\overline{23}|} = 16.7147$

Calculate the Full Preliminary Term reserve at time 2 for a 25-year fully discrete endowment insurance, issued to (40), with sum insured 75,000.

(A) 2,188 (B) 2,190 (C) 2,192 (D) 2,194 (E) 2,196

**** END OF EXAMINATION ****

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1. [Chapter 7] Answer: (B)

Let G be the gross premium. By the equivalence principle,

APV of gross premiums = APV of death benefit + APV of expenses.

$$\begin{aligned} G\ddot{a}_{[x]} &= 1010\bar{A}_{[x]} + 0.9G + 0.1Ga_{[x]} + 15 + 3a_{[x]} \\ &= 1010\bar{A}_{[x]} + 0.9G + 0.1G(\ddot{a}_{[x]} - 1) + 15 + 3(\ddot{a}_{[x]} - 1) \\ G &= \frac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8} \end{aligned}$$

2. [Chapter 8] Answer: (D)

$U = \dfrac{1.96\text{SE}[\hat{S}(t)]}{\hat{S}(t)\ln\hat{S}(t)} = \dfrac{1.96 \times 0.0378}{0.659\ln 0.659} = -0.26958$, and the upper limit is $0.659^{\exp(-0.26958)} = 0.659^{0.763697} = 0.727249$.

3. [Chapter 4] Answer: (E)

We first change all annuities into insurances.

Statement (ii) implies that $A_x = 1 - \frac{0.07}{1.07} \times 11.7089 = 0.2340$.

Statement (iii) implies that $A_{x:\overline{40}|} = 1 - \frac{0.07}{1.07} \times 11.55 = 0.2444$.

Statement (iv) implies that $A_{x+40} = 1 - \frac{0.07}{1.07} \times 7.1889 = 0.5297$.

Finally, by $A_x = A^{1}_{x:\overline{40}|} + A_{x:\overset{1}{\overline{40}|}}A_{x+40}$ and $A_{x:\overset{1}{\overline{40}|}} + A^{1}_{x:\overline{40}|} = A_{x:\overline{40}|}$,

$$0.2340 = A^{1}_{x:\overline{40}|} + (0.2444 - A^{1}_{x:\overline{40}|}) \times 0.5297$$

On solving, we get $A^{1}_{x:\overline{40}|} = \dfrac{0.2340 - 0.2444 \times 0.5297}{1 - 0.5297} = 0.2223$.

4. [Chapter 2] Answer: (E)

Method 1:

$$\begin{aligned} {}_{1.8}p_{[66]+0.6} &= {}_{0.4}p_{[66]+0.6} \times {}_{1.4}p_{[66]+1} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{[66]+2} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{68} \end{aligned}$$

Obviously, $p_{[66]+1} = 1 - 0.135 = 0.865$, ${}_{0.4}p_{68} = 1 - 0.4q_{68} = 1 - 0.4(0.16) = 0.936$.

Finally, ${}_{0.4}p_{[66]+0.6} = p_{[66]}/{}_{0.6}p_{[66]} = 0.88/(1 - 0.6 \times 0.12) = 0.948276$.

So, the answer is ${}_{1.8}p_{[66]+0.6} = 0.76776$.

Method 2: ${}_{1.8}p_{[66]+0.6} = l_{[66]+2.4}/l_{[66]+0.6}$

Without loss of generality, let $l_{[66]} = 100$. Then we have:

$l_{[66]+1} = 88$, $l_{[66]+2} = 88 \times 0.865 = 76.12$, $l_{[66]+2} = 76.12 \times 0.84 = 63.9408$.

So, $l_{[66]+2.4} = 0.4 \times 63.9408 + 0.6 \times 76.12 = 71.24832$,

$l_{[66]+0.4} = 0.6 \times 88 + 0.4 \times 100 = 92.8$, and ${}_{1.8}p_{[66]+0.6} = 0.767762$.

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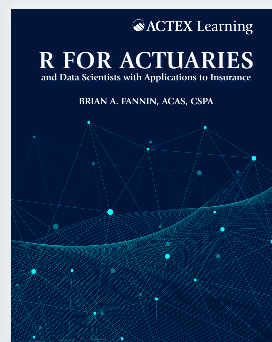

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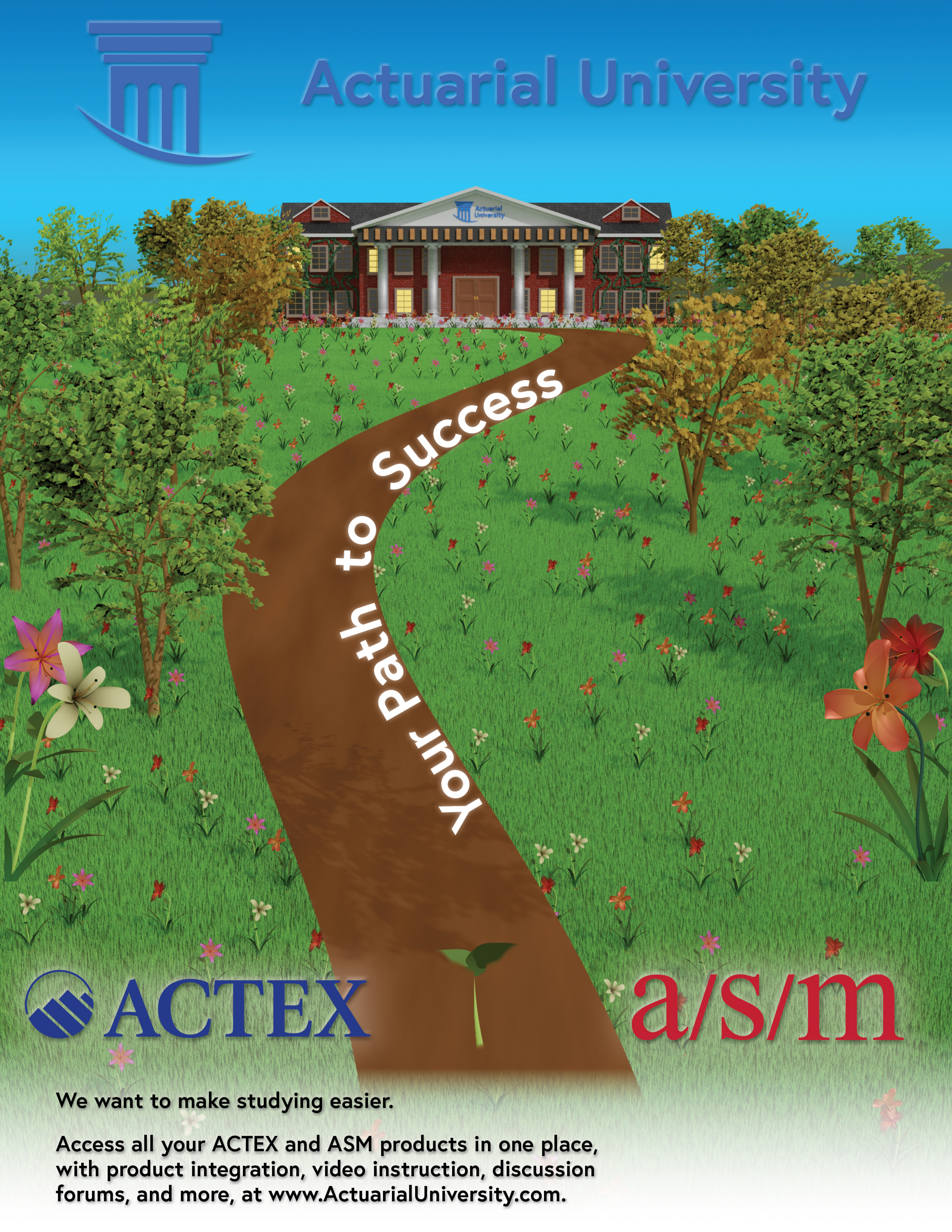
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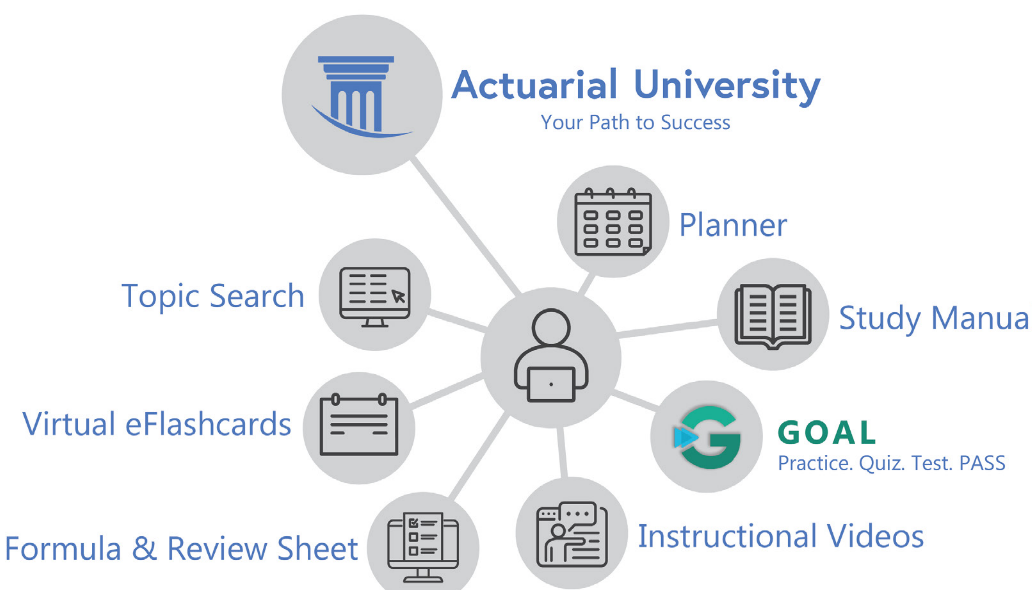

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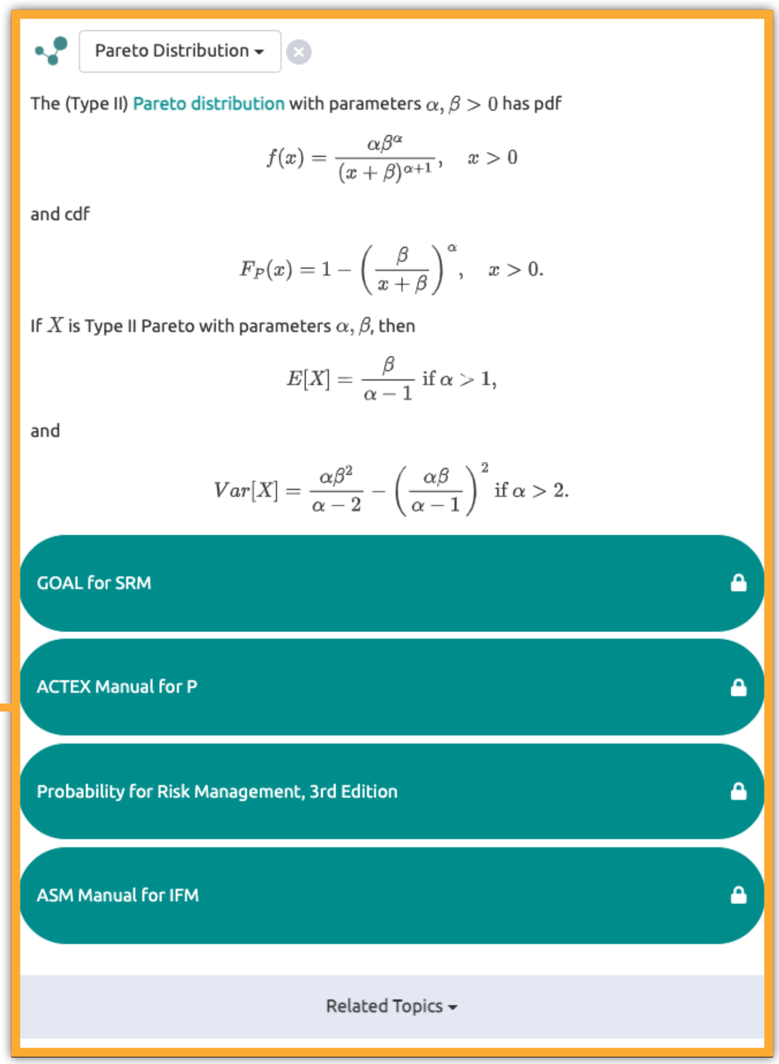

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INTRODUCTORY COMMENTS

The FAM exam is divided almost equally into FAM-S and FAM-L topics. This study guide is designed to help in the preparation for the Society of Actuaries FAM-S Exam.

The first part of this manual consists of a summary of notes, illustrative examples and problem sets with detailed solutions. The second part consists of 4 practice exams. The SOA exam syllabus for the FAM exam indicates that the exam is 3.5 hours in length with 40 multiple choice questions. The practice exams in this manual each have 20 questions, reflecting the fact that FAM-S is 50% of the full FAM exam. The appropriate time for the 20 question FAM-S practice exams in this manual is one hour and forty-five minutes.

The level of difficulty of the practice exam questions has been designed to be similar to those on past exams covering the same topics. The practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official references. I have been, perhaps, more thorough than necessary on reviewing maximum likelihood estimation.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that you have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into 32 sections of varying lengths, with some suggested time frames for covering the material. There are almost 180 examples in the notes and over 440 exercises in the problem sets, all with detailed solutions. The 4 practice exams have 20 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are almost 700 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study guide.

I suggest that you work through the study guide by studying a section of notes and then attempting the exercises in the problem set that follows that section. The order of the sections of notes is the order that I recommend in covering the material, although the material on pricing and reserving in Sections 27 to 31 is independent of the other material on the exam. The order of topics in this manual is not the same as the order presented on the exam syllabus.

It has been my intention to make this study guide self-contained and comprehensive for the FAM-S Exam topics, however it is important to be familiar with original reference material on all topics.

While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations.

In order for the review notes in this study guide to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study guide. The study guide begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance. Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org.

If you have any questions, comments, criticisms or compliments regarding this study guide, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from www.actexmadriver.com. It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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Section 7

Mixture Of Two Distributions

The material in this section relates to Section 4.2.3 of "Loss Models". The suggested time frame for this section is 2 hours. The topic of distribution mixtures is not mentioned in the learning objectives for the FAM exam but all of Chapter 4 of "Loss Models" is listed in the reference reading.

7.1 Mixture of Two Distributions

We begin with a formal algebraic definition of how a mixture of two distributions is constructed, and later we will look at how a mixture distribution is described by general reasoning.

Given random variables X_1 and X_2, with pdf's or pf's $f_{X_1}(x)$ and $f_{X_2}(x)$, and given $0 < a < 1$, we construct the random variable Y with pdf

$$\boldsymbol{f_Y(y) = a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)} \tag{7.1}$$

Y is called a **mixture distribution** or a two-point mixture of the distributions of X_1 and X_2.

The two-point mixture random variable Y can also be defined in terms of the cdf,

$$\boldsymbol{F_Y(y) = a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y)} \tag{7.2}$$

X_1 and X_2 are the **component distributions** of the mixture, and the factors a and $1-a$ are referred to as **mixing weights**. It is important to understand that we are **not adding** aX_1 and $(1-a)X_2$, Y **is not** a sum of random variables. Y is defined in terms of a pdf (or cdf) that is a weighted average of the pdf's (or cdf's) of X_1 and X_2. We are adding af_{X_1} and $(1-a)f_{X_2}$ to get f_Y.

Example 7.1. As a simple illustration of a mixture distribution, consider two bowls. Bowl A has 5 balls with the number 1 on them and 5 balls with the number 2 on them, and bowl B has 3 balls with the number 1 and 7 balls with the number 2. Let X_1 denote the number on a ball randomly chosen from bowl A, and let X_2 denote the number on a ball randomly chosen from bowl B. The probability functions of X_1 and X_2 are
$f_{X_1}(1) = f_{X_1}(2) = 0.5$ and $f_{X_2}(1) = 0.3$, $f_{X_2}(2) = 0.7$.

Suppose we create the mixture distribution with mixing weights $a = 0.5$ and $1 - a = 0.5$. The mixture distribution Y has probability function $f_Y(1) = 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4$, $f_Y(2) = 0.5 \times 0.5 + 0.5 \times 0.7 = 0.6$.

Note that the outcomes (ball numbers) of the mixture distribution Y come from the possible outcomes of the component distributions X_1 and X_2. □

An alternative way of looking at this mixture distribution is by means of conditioning on a "parameter". This will be important when we look at continuous mixing. The parameter approach to describe the mixture distribution in Example 7.1 is as follows.

Suppose that a fair coin is tossed. If the toss is a head, a ball is chosen at random from bowl A and if the toss is a tail, a ball is chosen at random from bowl B. We define the random variable Z to be the number on the ball. We will see that Z has the same distribution as the mixture distribution labeled Y above.

The random variable Z can be interpreted as follows. Consider the 2-point random variable Θ, for which $\Theta =$Bowl A if the coin toss is a head, and $\Theta =$Bowl B if the toss is a tail. Then $P(\Theta = A) = P(\Theta = B) = .5$ (since the coin is fair). Θ is used to indicate which bowl the ball will be chosen from depending on the outcome of the coin toss.

If the toss is a head, the bowl is A, and then Z has the X_1 distribution for the number on the ball, so $f_{X_1}(1) = P(Z = 1|\Theta = A) = 0.5$ and $f_{X_1}(2) = P(Z = 2|\Theta = A) = 0.5$. In a similar way, if the toss is a tail, the bowl is B, and then Z has the X_2 distribution for the number on the ball, so $f_{X_2}(1) = P(Z = 1|\Theta = B) = 0.3$ and $f_{X_2}(2) = P(Z = 2|\Theta = B) = 0.7$.
Z is described as a combination of two conditional distributions based on the parameter Θ.

To find the overall, or unconditional distribution of Z, we use some basic rules of probability. Since Θ must be A or B, we can think of bowl B as the "complement" of bowl A, and then

$$\begin{aligned} P(Z = 1) &= P[(Z = 1) \cap (\Theta = A)] + P[(Z = 1) \cap (\Theta = B)] \\ &= P(Z = 1|\Theta = A) \times P(\Theta = A) + P(Z = 1|\Theta = B) \times P(\Theta = B) \\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4 \end{aligned}$$

We have used the rule $P(C) = P[C \cap D] + P[C \cap D'] = P(C|D) \times P(D) + P(C|D') \times P(D')$.

This shows that the distribution of Z is the same as the mixture distribution Y in Example 7.1. The mixing weights for the two bowls are the probabilities of the coin indicating bowl A or bowl B.

Language used on exam questions that identifies a mixture distribution
There is some typical language that is used in exam questions that indicates that a mixture of distributions is being considered. This language will be illustrated using the distributions involved in the bowl example.

Suppose that we are told that there are two types of individuals. Type A individuals have a mortality probability of .5 (and survival probability of .5) in the coming year, and Type B individuals have a mortality probability of .3 in the coming year. In a large group of these individuals, 50% are Type A and 50% are Type B. An individual is chosen at random from the group. We want to find this randomly chosen individual's mortality probability.

We can use the usual rules of conditional probability to formulate this probability, just as above:

$$\begin{aligned} P(\text{dying this year}) &= P(\text{dying} \cap \text{Type A}) + P(\text{dying} \cap \text{Type B}) \\ &= P(\text{dying}|\text{Type A}) \times P(\text{Type A}) + P(\text{dying} \cap \text{Type B}) \times P(\text{Type B}) \\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4. \end{aligned}$$

This is exactly $f_Y(1) = f_{X_1}(1) \times a + f_{X_2}(1) \times (1-a)$ where "1" corresponds to the event of dying within the year. Note that the phrase "50% are Type A" is interpreted as meaning that if an individual is chosen from the large group, the probability of being Type A is 0.5. This is language that is often used in exam questions to indicate a mixture.

Mixture of a discrete and a continuous distribution

In Section 2 of this study guide we looked at "mixed distributions". In that section, a mixed distribution referred to a random variable that was continuous on part of its probability space and also had one or more discrete points. The concept of mixed distribution just introduced in this section can be used to describe the Section 2 type of mixed distribution. The following example uses the example from Section 2 to illustrate this.

Example 7.2. Suppose that X is defined in the following piecewise way: X has a discrete point of probability of .5 at $X = 0$, and on the interval (0,1) X is a continuous random variable with density function $f(x) = x$ for $0 < x < 1$, and X has no density or probability elsewhere. Show that this random variable can be described as a mixture of two distributions with appropriate definitions for component distributions X_1, X_2 and mixing weights a and $1-a$.

Solution.

Let $X_1 = 0$ be a constant (not actually a random variable, but is sometimes called a degenerate random variable), so that $f_{X_1}(0) = P(X_1 = 0) = 1$ and $f_{X_1}(x) = 0$ for $x \neq 0$.

Let X_2 be continuous on (0,1) with pdf $f_{X_2}(x) = 2x$.

With mixing weights of $a = 0.5$ and $1 - a = 0.5$, using the definition of the mixture of two distributions, we have the mixed random variable Y satisfying

$f_Y(0) = a \times f_{X_1}(0) + (1-a) \times f_{X_2}(0) = (0.5 \times 1) + 0 = 0.5$, and
$f_Y(x) = a \times f_{X_1}(x) + (1-a) \times f_{X_2}(x) = 0.5 \times 0 + 0.5 \times 2x = x$ for $0 < x < 1$.
Y has the same distribution as the original X. □

We should be a little careful about the situation in Example 7.2. When we are mixing discrete probability points with a continuous density, for any particular discrete point we always assign a probability of 0 at that point for any continuous component distribution. For instance, in Example 7.2, if X_1 was at the single point 0.4, then $f_{X_1}(0.4) = 1$ and $f_{X_1}(x) = 0$ elsewhere, and for the mixture distribution, then $f_Y(0.4) = 0.5 \times 1 = 0.5$ (we do not add $0.5 \times f_{X_2}(0.4)$).

Some important relationships for mixture distributions

We have already seen the defining relationships

$$\boldsymbol{f_Y(y) = af_{X_1}(y) + (1-a)f_{X_2}(y)}$$

and

$$\boldsymbol{F_Y(y) = aF_{X_1}(y) + (1-a)F_{X_2}(y)}.$$

We can interpret these relationships by saying that the pdf and cdf of Y are weighted averages of the component pdf's and cdf's. This weighted average interpretation can be applied to a number of other distribution related quantities.

- if C is any event related to Y, then $P(C) = a \times P_{X_1}(C) + (1-a) \times P_{X_2}(C)$; (7.3)
  $P_{X_1}(C)$ is the event probability based on random variable X_1, and the same for X_2
- If g is any function (that doesn't involve the parameters of Y), then (7.4)
  $E[g(Y)] = a \times E[g(X_1)] + (1-a) \times E[g(X_2)]$

The justification for this relationship follows from the form of the mixed density;

$$\begin{aligned} E[g(Y)] &= \int g(y)\, f_Y(y)\, dy = \int g(y) \times [a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)]\, dy \\ &= a \times \int g(y) \times f_{X_1}(y)\, dy + (1-a) \times \int g(y) \times f_{X_2}(y)\, dy \\ &= a \times E[g(X_1)] + (1-a) \times E[g(X_2)] \end{aligned}$$

Some of the important examples of these weighted average relationships are:

Interval Probability: the event C is $c < Y \le d$

$$\boldsymbol{P(c < Y \le d) = a \times P(c < X_1 \le d) + (1-a) \times P(c < X_2 \le d)} \tag{7.5}$$

kth moment of Y: the function g is $g(Y) = Y^k$

$$\boldsymbol{E[Y^k] = a \times E[X_1^k] + (1-a) \times E[X_2^k]} \tag{7.6}$$

7.2 Formulating a Mixture Distribution as a Combination of Conditional Distributions

The relationships above can also be formulated as applications of probability rules involving conditional probability and conditional expectation.

(i) For any random variable Y and event D,

$$\boldsymbol{f_Y(y) = f(y|D) \times P(D) + f(y|D') \times P(D')} \tag{7.7}$$

(ii) For any events C and D,

$$\boldsymbol{P[C] = P[C \cap D] + P[C \cap D'] = P[C|D] \times P[D] + P[C|D'] \times P[D'].} \tag{7.8}$$

(iii) For any random variable Y and any event D,

$$\boldsymbol{E[Y] = E[Y|D] \times P[D] + E[Y|D'] \times P[D']} \tag{7.9}$$

We can define the random variable Θ to be 1 or 2, indicating whether or not event D has occurred, so $\Theta = 1 \equiv D$ has occurred, and $\Theta = 2 \equiv D'$ has not occurred.

We can think of X_1 as the conditional distribution of Y given $\Theta = 1$ (event D) and we can think of X_2 as the conditional distribution of Y given $\Theta = 2$ (event D'), so that

$$f(y|D) = f(y|\Theta = 1) = f_{X_1}(y) \text{ and } f(y|D') = f(y|\Theta = 2) = f_{X_2}(y).$$

The cdf, pdf and expectation of Y are adaptations of the conditional probability rules above. For instance, the mean of the mixed distribution is the "mixture of the means" of the component distributions.

$$E[Y] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2] = E[X_1] \times a + E[X_2] \times (1-a).$$

Example 7.3. A collection of insurance policies consists of two types. 25% of policies are Type 1 and 75% of policies are Type 2. For a policy of Type 1, the loss amount per year follows an exponential distribution with mean 200, and for a policy of Type 2, the loss amount per year follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 200$. For a policy chosen at random from the entire collection of both types of policies, find the probability that the annual loss will be less than 100, and find the average loss.

Solution.

The two types of losses are the random variables X_1 and X_2.

X_1 has an exponential distribution with mean 200, so it has cdf $F_{X_1}(x) = 1 - e^{-x/200}$.

X_2 has cdf $F_{X_2}(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} = 1 - \left(\frac{200}{x+200}\right)^{3}$.

These are the component distributions. The mixing weights are the proportions of policies of each type, so that $a = 0.25$ (the proportion of Type 1 policies) and $1 - a = 0.75$. The loss random variable Y of the randomly chosen policy has a distribution that is the mixture of X_1 and X_2. In this context, "randomly chosen" is taken to mean that the probabilities of choosing a Type 1 or Type 2 policy are the proportions of those policy types in the full collection of policies.

The cdf of Y is

$$
\begin{aligned}
F_Y(y) &= a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y) \\
&= 0.25 \times (1 - e^{-y/200}) + 0.75 \times \left(1 - \left(\frac{200}{y+200}\right)^{3}\right).
\end{aligned}
$$

Then, $F_Y(100) = 0.626$.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$. □

7.3 The Variance of a Mixed Distribution IS (usually) NOT the Weighted Average of the Variances

For a mixture of two random variables, the weighted average pattern applies to most quantities for the mixed distribution.

One important exception to this rule is the formulation of the **variance of the mixture**. If Y is any random variable, then we can formulate the variance of Y as $Var[Y] = E[Y^2] - (E[Y])^2$.

If Y is the mixture of X_1 and X_2 with mixing weights a and $1-a$, then we know that
$E[Y] = a \times E[X_1] + (1-a) \times E[X_2]$ and $E[Y^2] = a \times E[X_1^2] + (1-a) \times E[X_2^2]$.

It is tempting to guess that $Var[Y]$ is the weighted average of $Var[X_1]$ and $Var[X_2]$.

This is **not true** in general (we will see a special case later for which it is true).

Example 7.4. Find the variance of the loss on the policy chosen at random in Example 7.3, and compare it to the weighted average of the variances of the two component loss distributions.

Solution.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$,

and the second moment of Y is

$0.25 \times E[X_1^2] + 0.75 \times E[X_2^2] = 0.25 \times 2 \times 200^2 + 0.75 \times \frac{200^2 \times 2}{(3-1)(3-2)} = 50{,}000$, so

$Var[Y] = E[Y^2] - (E[Y])^2 = 34{,}375$.

$Var[X_1] = 40{,}000$ (variance of an exponential distribution is the square of the mean).

$Var[X_2] = \frac{200^2 \times 2}{(3-1)(3-2)} - 100^2 = 30{,}000$.

The weighted average of the two variances is $0.25 \times 40{,}000 + 0.75 \times 30{,}000 = 32{,}500$, which is not the variance of Y. □

Weighted average does not apply to percentiles in a mixture distribution
In Example 7.3, the 50th percentile of X_1 is m_1, the solution of the equation $F_{X_1}(m_1) = 1 - e^{-m_1/200} = 0.5$, so that $m_1 = 138.63$.

The 50th percentile of X_2 is m_2, the solution of $F_{X_2}(m_2) = 1 - \left(\frac{200}{m_2+200}\right)^3 = 0.5$, so that $m_2 = 51.98$.

The weighted average of the two 50th percentiles is $0.25 \times 138.63 + 0.75 \times 51.98 = 73.64$. The cdf of the mixed distribution Y is $F_Y(y) = 0.25 \times F_{X_1}(y) + 0.75 \times F_{X_2}(y)$, and $F_Y(73.64) = 0.53$, so 73.64 is not the 50th percentile of Y.

To find the 50th percentile, say m, of the mixed distribution in Example 7.3, we must solve the equation

$$\begin{aligned} .50 = F_Y(m) &= 0.25 \times F_{X_1}(m) + 0.75 \times F_{X_2}(m) \\ &= 0.25 \times (1 - e^{-m/200}) + 0.75 \times \left(1 - \left(\frac{200}{m+200}\right)^3\right). \end{aligned}$$

There is no algebraic solution, and m would have to be found by a numerical approximation method. The 50th percentile turns out to be about 65.7.

Section 7 Problem Set

Mixture Of Two Distributions

1. A portfolio of insurance policies is divided into low and high risk policies. 80% of the policies are low risk and the rest are high risk. The annual number of claims on a low risk policy has a Poisson distribution with a mean of .25 and the annual number of claims on a high risk policy has a Poisson distribution with a mean of 2. A policy is randomly chosen from the portfolio.

   (a) Find the mean and variance of the number of annual claims for the policy.

   (b) Find the probability that the policy has at most 1 claim in the coming year.

2. The distribution of a loss, X, is a two-point mixture:

   (i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

   (ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

   Calculate $Pr(X \leq 200)$.

   (A) 0.76 (B) 0.79 (C) 0.82 (D) 0.85 (E) 0.88

3. The random variable N has a mixed distribution:

   (i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.

   (ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

   Which of the following is a correct expression for Prob($N = 2$)?

   (A) $0.125p^2$
   (B) $0.375 + 0.125p$
   (C) $0.375 + 0.125p^2$
   (D) $0.375 - 0.125p^2$
   (E) $0.375 - 0.125p$

4. Y is a mixture of two exponential distributions, $f_Y(y) = \frac{1}{2}e^{-y} + \frac{1}{6}e^{-y/3}$.
   The random variable Z defined by the equation $Z = 2Y$.
   Z is a mixture of two exponentials. The means of those two exponential distributions are

   (A) 1 and 3
   (B) 1 and 6
   (C) 2 and 3
   (D) 2 and 6
   (E) 3 and 6

5. X_1 has a uniform distribution on the interval (0,1000) and X_2 has a uniform distribution on the interval (0,2000). Y is defined as a mixture of X_1 and X_2 with mixing weights of .5 for each mixture component. Find the pdf, cdf and median (50th percentile) of Y.

6. A population of people aged 50 consists of twice as many non-smokers as smokers. Non-smokers at age 50 have a mortality probability of .1 and smokers at age 50 have a mortality probability of .2. Two 50-year old individuals are chosen at random from the population.

   (a) Find the probability that at least one of them dies before age 51.

   (b) Suppose that the mortality probabilities for smokers and non-smokers remain the same at age 51. Find the mortality probability of a randomly chosen survivor at age 51 in this population.

7. Y is the mixture of an exponential random variable with mean 1 and mixing weight $\frac{2}{3}$, and an exponential distribution with mean 2 and mixing weight $\frac{1}{3}$.

   Find the pdf, cdf, mean, variance and 90th percentile of Y.

8. Which of the following statements are true?

   I. A mixture of two different exponentials with the mixture having a mean 2 has a heavier right tail than a single exponential distribution with mean 2.

   II. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y)$, where $0 < a_1 < 1$ and $0 < a_2 < 1$, then $e_Y(d) = a_1 \times e_{X_1}(d) + a_2 \times e_{X_2}(d)$.

9. You are given two independent estimates of an unknown quantity, μ:

   (i) Estimate A: $E(\mu_A)$= 1000 and $\sigma(\mu_A)$= 400

   (ii) Estimate B: $E(\mu_B)$= 1200 and $\sigma(\mu_B)$= 200

   Estimate C is a weighted average of the two estimates A and B, such that:

$$\mu_C = w \times \mu_A + (1 - w) \times \mu_B$$

   Determine the value of w that minimizes $\sigma(\mu_C)$.

   (A) 0 (B) 1/5 (C) 1/4 (D) 1/3 (E) 1/2

10. You are given the claim count data for which the sample mean is roughly equal to the sample variance. Thus you would like to use a claim count model that has its mean equal to its variance. An obvious choice is the Poisson distribution. Determine which of the following models may also be appropriate.

   (A) A mixture of two binomial distributions with different means.

   (B) A mixture of two Poisson distributions with different means.

   (C) A mixture of two negative binomial distributions with different means.

   (D) None of A, B, or C

   (E) All of A, B, and C

11. Losses come from an equally weighted mixture of an exponential distribution with mean m_1, and an exponential distribution with mean m_2. Determine the least upper bound for the coefficient of variation of this distribution.

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) $\sqrt{5}$

12. X has the following distribution : $Pr[X = 0] = 0.4$, $Pr[X = 1] = 0.6$.
The distribution of Y is conditional on the value of X: if $X = 0$, then the distribution of Y is $Pr[Y = 0] = 0.6$, $Pr[Y = 1] = 0.2$, $Pr[Y = 2] = 0.2$, and if $X = 1$, then the distribution of Y is $Pr[Y = 0] = 0.2$,$Pr[Y = 1] = 0.3$, $Pr[Y = 2] = 0.5$.
Z is the sum of Y independent normal random variables, each with mean and variance 2.

What is $Var[Z]$?

(A) 5.0 (B) 6.0 (C) 7.0 (D) 8.0 (E) 9.0

13. Subway trains arrive at your station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types and number of trains arriving are independent. An express gets you to work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You are both waiting at the same station.

Calculate the conditional probability that you arrive at work before your co-worker, given that a local arrives first.

(A) 37% (B) 40% (C) 43% (D) 46% (E) 49%

Section 7 Problem Set Solutions

1. (a) Y is the mixture of two Poisson distributions.
$E[Y] = 0.8 \times E[X_L] + 0.2 \times E[X_H] = 0.8 \times 0.250 + 0.2 \times 2 = 0.6$.
To find the second moment of Y, we recall that for a Poisson distribution with mean λ, the variance is also λ. Since $Var[X] = E[X^2] - (E[X])^2$, it follows that
$E[X^2] = Var[X] + (E[X])^2 = \lambda + \lambda^2$ for a Poisson distribution.

Then, $E[Y^2] = 0.8 \times E[X_L^2] + 0.2 \times E[X_H^2] = 0.8 \times (0.25 + 0.25^2) + 0.2 \times (2 + 2^2) = 1.45$.
Then $Var[Y] = 1.45 - (0.6)^2 = 1.09$.

(b) $$P(Y \le 1) = 0.8 \times P(X_1 \le 1) + 0.2 \times P(X_2 \le 1)$$
$$= 0.8 \times [e^{-0.25} + 0.25e^{-0.25}] + 0.2 \times [e^{-2} + 2e^{-2}] = 0.860.$$

2. The probability is a mixture of the probabilities for the two components.
$P(X \le 200) = 0.8 \times P(X_1 \le 200) + 0.2 \times P(X_2 \le 200)$, where X_1 and X_2 are the two Pareto distributions.

$P(X_1 \le 200) = 1 - \left(\frac{100}{200+100}\right)^2 = .8889$, and
$P(X_2 \le 200) = 1 - \left(\frac{3000}{200+3000}\right)^4 = .2275$.
$P(X \le 200) = (0.8 \times 0.8889) + (0.2 \times 0.2275) = 0.757$.

Answer A

3. $$P(N = 2) = p \times P(N_1 = 2) + (1 - p) \times P(N_2 = 2)$$
$$= p \times (0.5)^2 + (1 - p) \times 6 \times (0.5)^4 = 0.375 - 0.125p.$$

We have used the binomial probabilities of the form $\binom{m}{k} q^k (1-q)^{m-k}$.

Answer E

4. $f_Y(y) = \frac{1}{2} \times e^{-y} + \frac{1}{2} \times \frac{1}{3} e^{-y/3}$. $F_Y(y) = \frac{1}{2} \times (1 - e^{-y}) + \frac{1}{2} \times (1 - e^{-y/3})$.
$F_Z(z) = F_Y(\frac{z}{2}) = \frac{1}{2} \times (1 - e^{-z/2}) + \frac{1}{2} \times (1 - e^{-z/6})$.
Z is a mixture of two exponentials with means 2 and 6.

Answer D

5. $$f_Y(y) = 0.5 \times f_{X_1}(y) + 0.5 \times f_{X_2}(y)$$
$$= \begin{cases} 0.5 \times 0.001 + 0.5 \times 0.0005 = 0.00075 & \text{if } 0 < y < 1000 \\ 0.5 \times 0.0005 = 0.00025 & \text{if } 1000 \le y < 2000 \end{cases}$$
$$F_Y(y) = 0.5 \times F_{X_1}(y) + 0.5 \times F_{X_2}(y)$$
$$= \begin{cases} 0.5 \times 0.001x + 0.5 \times 0.0005x = 0.00075x & \text{if } 0 < y < 1000 \\ 0.5 \times 1 + 0.5 \times 0.0005x = 0.5 + 0.00025x & \text{if } 1000 \le y < 2000 \end{cases}$$

The median of Y, m, must satisfy the equation $F_Y(m) = 0.5$.
We see that at $y = 1000$, $F_Y(1000) = 0.75$. Therefore, $m < 1000$, so that
$F_Y(m) = 0.00075m = 0.5$. Therefore, $m = 666.67$.
Note that the mean of Y is $0.5 \times 500 + 0.5 \times 1000 = 750$.

6. For a randomly chosen individual at age 50, the mortality probability is the mixture of the mortality probabilities for non-smokers and smokers. This is $q = \frac{2}{3} \times 0.1 + \frac{1}{3} \times 0.2 = \frac{2}{15}$.

   The probability that both of two independent 50-year old individuals survive the year is $(1 - \frac{2}{15})^2 = 0.7511$. The probability at least one of them dies by age 51 is $1 - 0.7511 = 0.25$.

   Suppose that there are 1000 non-smokers and 500 smokers at age 50.
   The expected number of surviving non-smokers at age 51 is $1000 \times 0.9 = 900$, and the number of surviving smokers is $500 \times 0.8 = 400$.

   A randomly chosen survivor at age 51 has a $\frac{9}{13}$ chance of being a non-smoker and a $\frac{4}{13}$ chance of being a smoker.

   The mortality probability at age 51 for the randomly chosen survivor is
   $\frac{9}{13} \times 0.1 + \frac{4}{13} \times 0.2 = .131$.

7. $f_Y(y) = \frac{2}{3} \times e^{-y} + \frac{1}{3} \times \frac{1}{2} e^{-y/2}$.

   $F_Y(y) = \frac{2}{3} \times (1 - e^{-y}) + \frac{1}{3} \times (1 - e^{-y/2})$.

   $E[Y] = \frac{2}{3} \times 1 + \frac{1}{3} \times 2 = \frac{4}{3}$.

   $E[Y^2] = \frac{2}{3} \times (2 \times 1^2) + \frac{1}{3} \times (2 \times 2^2) = 4 \to Var[Y] = 4 - (\frac{4}{3})^2 = \frac{20}{9}$.

   The 90th percentile of Y is c, which must satisfy the equation
   $F_Y(c) = \frac{2}{3} \times (1 - e^{-c}) + \frac{1}{3} \times (1 - e^{-c/2}) = 0.9$.

   If we let $r = e^{-c/2}$, then this becomes a quadratic equation in r,
   $\frac{2}{3} \times (1 - r^2) + \frac{1}{3} \times (1 - r) = 0.9$, or equivalently, $2r^2 + r - .3 = 0$.

   Solving for r results in $r = .21$ or $r = -0.71$.

   We ignore the negative root, since $r = e^{-c/2}$ must be positive.

   Then $c = -2 \ln r = -2 \ln .21 = 3.12$ is the 90th percentile of Y.

8. I. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y) = a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}$, and $a_1\theta_1 + a_2\theta_2 = 2$. Then $\theta_1 < 2 < \theta_2$ (or vice-versa).

   For an exponential distribution with mean 2, we have $f_Z(y) = \frac{1}{2} e^{-y/2}$
   $\frac{f_Y(y)}{f_Z(y)} = \frac{a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}}{\frac{1}{2} e^{-y/2}} = \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})}$.

   Since $\theta_1 < 2 < \theta_2$, it follows that $\frac{1}{\theta_2} < \frac{1}{2} < \frac{1}{\theta_1}$. Therefore, $\frac{1}{2} - \frac{1}{\theta_2} > 0$
   so that $\lim_{y \to \infty} \frac{f_Y(y)}{f_Z(y)} = \lim_{y \to \infty} \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})} = 0 + \infty$. True.

   II. Suppose that $f_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{6} e^{-y/3}$. Then $S_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{2} e^{-y/3}$, so that Y is mixture of exponentials with means of 1 and 3, with equal mixing weights of $\frac{1}{2}$.
   $e_Y(d) = \frac{\int_d^\infty S(y)\,dy}{S(d)} = \frac{\frac{1}{2} e^{-d} + \frac{3}{2} e^{-d/3}}{\frac{1}{2} e^{-d} + \frac{1}{2} e^{-d/3}}$.
   $e_{X_1}(d) = 1$, $e_{X_3}(d) = 3 \ \to \ \frac{1}{2} \times e_{X_1}(d) + \frac{1}{2} \times e_{X_3}(d) = 2$. False.

9. Since the two estimates are independent,

$$Var[\mu_C] = Var[w \times \mu_A + (1-w) \times \mu_B] = w^2 \times Var[\mu_A] + (1-w)^2 \times Var[\mu_B]$$
$$= w^2 \times 400^2 + (1-w)^2 \times 200^2 = 200{,}000w^2 - 80{,}000w + 40{,}000.$$

This is a quadratic expression in w with positive coefficient of w^2. The minimum can be found by differentiating with respect to w and setting equal to 0:
$400{,}000w - 80{,}000 = 0 \rightarrow w = \frac{1}{5}$ is the value of w that minimizes $Var[\mu_c]$.

Answer B

10. If Y is a mixture of X_1 and X_2 with mixing weights a and $1-a$, we can define the parameter $\Theta = \{1,2\}$, with $P[\Theta = 1] = a$, $P[\Theta = 2] = 1-a$.
Then $E[Y] = E[\,E[Y|\Theta]\,] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2]$
$= E[X_1] \times a + E[X_2] \times (1-a)$
(this is the usual way the mean of a finite mixture is formulated).

$Var[Y] = E[\,Var[Y|\Theta]\,] + Var[\,E[Y|\Theta]\,]$, where
$E[\,Var[Y|\Theta]\,] = Var[X_1] \times a + Var[X_2] \times (1-a)$ and
$Var[\,E[Y|\Theta]\,] = (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

The last equality follows from the fact that if Z is a two-point random variable
$Z = \begin{cases} u & \text{prob.}\, p \\ v & \text{prob.}\, 1-p \end{cases}$, then $Var[Z] = (u-v)^2 \times p \times (1-p)$; $E[Y|\Theta]$ is
a two-point random variable $E[Y|\Theta] = \begin{cases} E[X_1] & \text{prob.}\, a \\ E[X_2] & \text{prob.}\, 1-a \end{cases}$.
Therefore $Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

B) If X_1,X_2 are Poisson random variables then $E[X_1] = Var[X_1]$ and $E[X_2] = Var[X_2]$, so that $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) = Var[X_1] \times a + Var[X_2] \times (1-a)$.

It follows that
$Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a) > E[Y]$.

C) For a negative binomial random variable with parameters r and β, the mean is $r\beta$ and the variance is $r\beta(1+\beta)$, so the variance is larger than the mean. If X_1 and X_2 have negative binomial distributions, the $E[X_1] < Var[X_1]$ and $E[X_2] < Var[X_2]$.

Therefore, $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) < Var[X_1] \times a + Var[X_2] \times (1-a)$, and
$Var[X_1] \times a + Var[X_2] \times (1-a)$
$< Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a(1-a) = Var[Y]$;
therefore $E[Y] < Var[Y]$.

A) For a binomial random variable with parameters m and q, the mean is mq and the variance is $mq(1-q)$, which is smaller than the mean.

Therefore $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) > Var[X_1] \times a + Var[X_2] \times (1-a)$, and it is possible that when we add $(E[X_1] - E[X_2])^2 \times a \times (1-a)$ to the right side, we get approximate equality.
So it is possible that $E[Y] = Var[Y]$ for a mixture of binomials.

Answer A

11. The mean will be $\frac{1}{2}\times(m_1+m_2)$ and the variance will be $\frac{1}{2}\times(2m_1^2+2m_2^2)-[\frac{1}{2}\times(m_1+m_2)]^2$. The coefficient of variation is the ratio of standard deviation to mean.
The square of the coefficient of variation is

$$\begin{aligned}\frac{\frac{1}{2}\times(2m_1^2+2m_2^2)-[\frac{1}{2}\times(m_1+m_2)]^2}{[\frac{1}{2}\times(m_1+m_2)]^2} &= \frac{3m_1^2-2m_1m_2+3m_2^2}{(m_1+m_2)^2}\\ &= \frac{3m_1^2+6m_1m_2+3m_2^2-8m_1m_2}{(m_1+m_2)^2}\\ &= 3-\frac{8m_1m_2}{(m_1+m_2)^2}.\end{aligned}$$

The maximum square of the coefficient of variation is 3, and it occurs at the minimum of $\frac{8m_1m_2}{(m_1+m_2)^2}$, which is 0 (if m_1 or m_2 is 0).
The least upper bound of the coefficient of variation is $\sqrt{3}$.

Answer C

12. $Var[Z]=Var[\,E[Z\mid Y]\,]+E[\,Var[Z\mid Y]\,]$.

$E[Z\mid Y]=2\times Y$ (a sum of Y independent normal random variables each with mean 2), and $Var[Z\mid Y]=2\times Y$ (a sum of Y independent normal random variables each with a variance of 2).

Thus, $Var[Z]=Var[2\times Y]+E[2\times Y]=4\times Var[Y]+2\times E[Y]$.
$Var[Y]=Var[\,E[Y\mid X]\,]+E[\,Var[Y\mid X]\,]$.
Since $E[Y]=0.6$ with prob. $0.4\,(X=0)$ and $E[Y]=1.3$ with prob. $0.6\,(X=1)$, it follows that $Var[E[Y\mid X]]=(0.6)^2\times0.4+(1.3)^2\times0.6-(1.02)^2=0.1176$.

Also, if $X=0$, the variance of Y is $1^2\times0.2+2^2\times0.2-(0.6)^2=0.64$,
and if $X=1$, the variance of Y is $1^2\times0.3+2^2\times0.5-(1.3)^2=0.61$. Thus,
$E[Var[Y\mid X]]=0.64\times0.4+.61\times.6=0.622$, and so $Var[Y]=0.1176+0.622=0.7396$.

Then, the variance of Z is $4\times0.7396+2\times1.02=4.9984$. Note that $Var[Y]$ can also be found from $E[Y^2]-(E[Y])^2$. Again, $E[Y]=1.02$, as above, and now,
$E[Y^2]=E[E[Y^2\mid X]]=1\times0.4+2.3\times0.6=1.78$, so that $Var[Y]=0.7396$.

Answer A

13. Given that a local train arrives first, you will get to work 28 minutes after that local train arrives, since you will take it. Your co-worker will wait for first express train. You will get to work before your co-worker if the next express train (after the local) arrives more than 12 minutes after the local. We expect 5 express trains per hour, so the time between express trains is exponentially distributed with a mean of $\frac{1}{5}$ of an hour, or 12 minutes. Because of the lack of memory property of the exponential distribution, since we are given that the next train is local, the time until the next express train after that is exponential with a mean of 12 minutes. Therefore, the probability that after the local, the next express arrives in more than 12 minutes is $P[T>12]$, where T has an exponential distribution with a mean of 12. This probability is $e^{-12/12}=e^{-1}=0.368$ (37%).

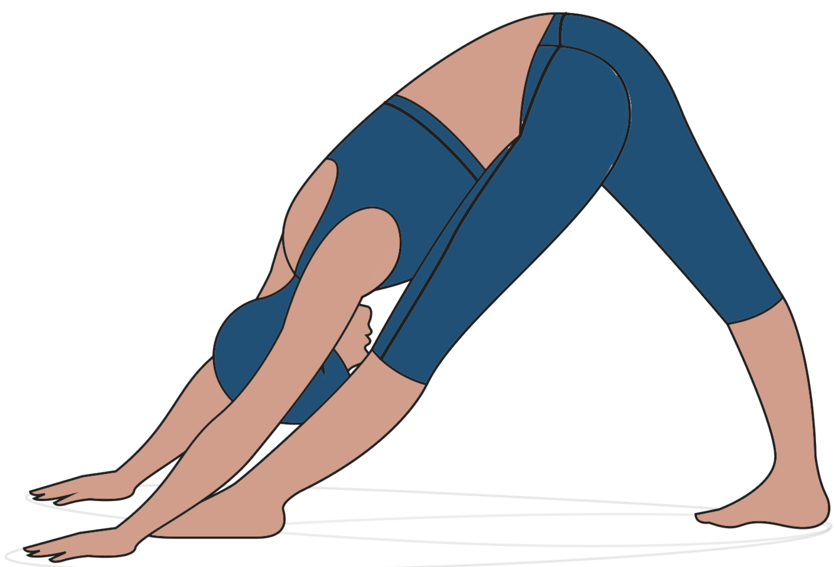
Answer A

STUDY BREAK

Do you have any fun ideas for future Study Breaks?

Share your inspiration with **#actexstudybreak**

Exercise is a proven way to boost your verbal memory, thinking, and learning. [1]



Pyramid with Arms Extended

Yoga - Sun Salutations

*Hold each pose for three long breaths, breathing in through your nose and out through your mouth.

Note that we are mathematicians, not yogi: attempt poses at your own risk!

[1]: https://www.health.harvard.edu/blog/regular-exercise-changes-brain-improve-memory-thinking-skills-201404097110

Practice Exam 1

1. Each of decision-makers X, Y, and Z has the opportunity to participate in a game with payoff uniformly distributed on (0, 10000). Assume that X, Y and Z value assets of amount $w \geq 0$ according to the following utility functions:

Decision Maker	Utility Function
X	$u_X(w) = \sqrt{w}$
Y	$u_Y(w) = \frac{w}{100}$
Z	$u_Z(w) = \left(\frac{w}{100}\right)^2$

Which decision maker would not be willing to pay more than 5,000 to participate in the game?

(A) X and Y only

(B) X and Z only

(C) Y and Z only

(D) X, Y and Z

(E) The correct answer is not given by (A), (B), (C) or (D)

2. The XYZ Insurance Company sells property insurance policies with a deductible of \$5,000, policy limit of \$500,000, and a coinsurance factor of 80%. Let X_i be the individual loss amount of the ith claim and Y_i be the claim payment of the ith claim. Which of the following represents the relationship between X_i and Y_i?

(A) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 625{,}000 \\ 500{,}000 & X_i > 625{,}000 \end{cases}$$

(B) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 4{,}000) & 4{,}000 < X_i \leq 500{,}000 \\ 500{,}000 & X_i > 500{,}000 \end{cases}$$

(C) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 630{,}000 \\ 500{,}000 & X_i > 630{,}000 \end{cases}$$

(D) $$Y_i = \begin{cases} 0 & X_i \leq 6{,}250 \\ 0.80\,(X_i - 6{,}250) & 6{,}250 < X_i \leq 631{,}500 \\ 500{,}000 & X_i > 631{,}500 \end{cases}$$

(E) $$Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 505{,}000 \\ 500{,}000 & X_i > 505{,}000 \end{cases}$$

3. A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts can be any non-negative integer, 1, 2, 3, ... without limit. The probability that any given payout is equal to $i > 0$ is $\frac{1}{2^i}$. Payouts are independent. Calculate the probability that there are no payouts of 1,2, or 3 in a given 20 minute period.

(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

4. Zoom Buy Tire Store, a nationwide chain of retail tire stores, sells 2,000,000 tires per year of various sizes and models. Zoom Buy offers the following road hazard warranty: "If a tire sold by us is irreparably damaged in the first year after purchase, we'll replace it free, regardless of the cause."

The average annual cost of honoring this warranty is \$10,000,000, with a standard deviation of \$40,000. Individual claim counts follow a binomial distribution, and the average cost to replace a tire is \$100. All tires are equally likely to fail in the first year, and tire failures are independent. Calculate the standard deviation of the replacement cost per tire.

(A) Less than \$60
(B) At least \$60, but less than \$65
(C) At least \$65, but less than \$70
(D) At least \$70, but less than \$75
(E) At least \$75

5. A compound Poisson claim distribution has $\lambda = 3$ and individual claims amounts distributed as follows:

x	$f_X(x)$
5	0.6
10	0.4

Determine the expected cost of an aggregate stop-loss insurance with a deductible of 6.

(A) Less than 15.0
(B) At least 15.0 but less than 15.3
(C) At least 15.3 but less than 15.6
(D) At least 15.6 but less than 15.9
(E) At least 15.9

Use the following information for questions 6 and 7. You are the producer of a television quiz show that gives cash prizes. The number of prizes, N, and prize amounts, X are independent of one another and have the following distributions:

N: $P[N=1]=0.8, \quad P[N=2]=0.2$

X: $P[X=0]=0.2, \quad P[X=100]=0.7, \quad P[X=1000]=0.1$

6. Your budget for prizes equals the expected prizes plus 1.25× standard deviation of prizes. Calculate your budget.

(A) 384 (B) 394 (C) 494 (D) 588 (E) 596

7. You buy stop-loss insurance for prizes with a deductible of 200. The cost of insurance includes a 175% relative security load (the relative security load is the percentage of expected payment that is added). Calculate the cost of the insurance.

(A) 204 (B) 227 (C) 245 (D) 273 (E) 357

8. An actuary determines that claim counts follow a negative binomial distribution with unknown β and r. It is also determined that individual claim amounts are independent and identically distributed with mean 700 and variance 1,300. Aggregate losses have a mean of 48,000 and variance 80 million. Calculate the values for β and r.

(A) $\beta = 1.20, r = 57.19$
(B) $\beta = 1.38, r = 49.75$
(C) $\beta = 2.38, r = 28.83$
(D) $\beta = 1{,}663.81, r = 0.04$
(E) $\beta = 1{,}664.81, r = 0.04$

9. Let X_1, X_2, X_3 be independent Poisson random variables with means $\theta, 2\theta$, and 3θ respectively. What is the maximum likelihood estimator of θ based on sample values x_1, x_2, and x_3 from the distributions of X_1, X_2 and X_3, respectively,

(A) $\dfrac{\bar{x}}{2}$
(B) $\bar{x}$
(C) $\dfrac{x_1 + 2x_2 + 3x_3}{6}$
(D) $\dfrac{3x_1 + 2x_2 + x_3}{6}$
(E) $\dfrac{6x_1 + 3x_2 + 2x_3}{11}$

10. Loss random variable X has a uniform distribution on $(0, \theta)$. A sample is taken of n insurance payments from policies with a limit of 100. Eight of the sample values are limit payments of 100. The maximum likelihood estimate of θ is $\widehat{\theta}$. Another sample is taken, also of n insurance payments, but from policies with a limit of 150. Three of the sample values are limit payments of 150. The maximum likelihood estimate of θ is $\frac{4}{3}\widehat{\theta}$.

Determine n.

(A) 40 (B) 42 (C) 44 (D) 46 (E) 48

11. For a group of policies, you are given:

(i) Losses follow a uniform distribution on the interval $(0, \theta)$, where $\theta > 25$.

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
$x \leq 10$	n_1
$10 < x \leq 25$	n_2
$x > 25$	n_3

The maximum likelihood estimate of θ can be written in the form $25 + y$. Determine y.

(A) $\dfrac{25n_1}{n_2 + n_3}$

(B) $\dfrac{25n_2}{n_1 + n_3}$

(C) $\dfrac{25n_3}{n_1 + n_2}$

(D) $\dfrac{25n_1}{n_1 + n_2 + n_3}$

(E) $\dfrac{25n_2}{n_1 + n_2 + n_3}$

12. The number of claims follows a negative binomial distribution with parameters β and r, where β is unknown and r is known. You wish to estimate β based on n observations, where $\bar{x}$ is the mean of these observations. Determine the maximum likelihood estimate of β.

(A) $\dfrac{\bar{x}}{r^2}$ (B) $\dfrac{\bar{x}}{r}$ (C) $\bar{x}$ (D) $r\bar{x}$ (E) $r^2\bar{x}$

13. The following 6 observations are assumed to come from the continuous distribution with pdf $f(x;\theta) = \frac{1}{2}x^2\theta^3 e^{-\theta x}$: 1, 3, 4, 4, 5, 7.

Find the mle of θ.

(A) 0.25 (B) 0.50 (C) 0.75 (D) 1.00 (E) 1.25

14. An analysis of credibility premiums is being done for a particular compound Poisson claims distribution, where the criterion is that the total cost of claims is within 5% of the expected cost of claims with a probability of 90%. It is found that with $n = 60$ exposures (periods) and $\bar{X} = 180.0$, the credibility premium is 189.47. After 20 more exposures (for a total of 80) and revised $\bar{X} = 185$, the credibility premium is 190.88. After 20 more exposures (for a total of 100) the revised $\bar{X}$ is 187.5. Assuming that the manual premium remains unchanged in all cases, and assuming that full credibility has not been reached in any of the cases, find the credibility premium for the 100 exposure case.

(A) 191.5 (B) 192.5 (C) 193.5 (D) 194.5 (E) 196.5

15. For an insurance portfolio, you are given:

(i) For each individual insured, the number of claims follows a Poisson distribution.

(ii) The mean claim count varies by insured, and the distribution of mean claim counts follows gamma distribution.

(iii) For a random sample of 1000 insureds, the observed claim counts are as follows:

Number of Claims, n	0	1	2	3	4	5
Number of Insureds, f_n	512	307	123	41	11	6

$$\sum n f_n = 750, \qquad \sum n^2 f_n = 1494$$

(iv) Claim sizes follow a Pareto distribution with mean 1500 and variance 6,750,000.

(v) Claim sizes and claim counts are independent.

(vi) The full credibility standard is to be within 5% of the expected aggregate loss 95% of the time.

Determine the minimum number of insureds needed for the aggregate loss to be fully credible.

(A) Less than 8300

(B) At least 8300, but less than 8400

(C) At least 8400, but less than 8500

(D) At least 8500, but less than 8600

(E) At least 8600

Information on Questions 16 and 17 is as follows. You are given the following information on cumulative incurred losses through development years shown.

	Cumulative Incurred Losses				Paid-to-Date
Accident	Development Year				at Dec 31, AY4
Year	0	1	2	3	
AY1	2325	3749	4577	4701	4701
AY2	2657	4438	5529		4500
AY3	2913	4995			3500
AY4	3163				2500

16. Using an average factor model, calculate the estimated total loss reserve as of Dec. 31, AY4.

(A) Less than 7500

(B) At least 7500 but less than 7800

(C) At least 7800 but less than 8100

(D) At least 8100 but less than 8400

(E) At least 8400

17. As of Dec. 31, AY4, calculate

Estimated reserve for AY2 based on an average factor model)
− (Estimated reserve for AY2 based on a mean factor model)

(A) Less than −300

(B) At least −300 but less than −100

(C) At least −100 but less than 100

(D) At least 100 but less than 300

(E) At least 300

18. For a one-period binomial model for the price of a stock with price 100 at time 0, you are given:

(i) The stock pays no dividends.

(ii) The stock price is either 110 or 95 at the end of the year.

(iii) The risk free force of interest is 5%.

Calculate the price at time 0 of a one-year call option with strike price 100.

(A) Less than 6.00

(B) At least 6.00 but less than 6.25

(C) At least 6.25 but less than 6.50

(D) At least 6.50 but less than 6.75

(E) At least 6.75

19. Using the following information, determine the incurred losses for the 2017 accident year as reported at Dec. 31, 2018.

Occurrence #1: Occurrence date Feb. 1/16, Report date Apr. 1/16

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Apr. 1/16	1000	1000	2000
Dec. 31/16	1500	1000	2500
Dec. 31/17	1500	1000	2500
Mar. 1/18	3000	0	3000

Occurrence #2: Occurrence date May 1/17, Report date July 1/17

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
July 1/17	1000	2000	3000
Dec. 31/17	3000	1000	4000
Dec. 31/18	5000	0	5000

Occurrence #3: Occurrence date Nov. 1/17, Report date Feb. 1/18

Loss History

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Mar. 1/18	0	8000	8000
Dec. 31/18	5000	5000	10,000

(A) 0
(B) 3,000
(C) 5,000
(D) 8,000
(E) 15,000

20. You are given the following calendar year earned premium.

Year	CY2	CY3	CY4
Earned Premium	4200	4700	5000

You are also given the following rate changes

Date	April 1, CY1	September 1, CY2	July 1, CY3
Average Rate Change	+12 %	+6 %	+10 %

Determine the approximate earned premium at current (end of CY4) rates for CY3.

(A) Less than 5000
(B) At least 5000 but less than 5100
(C) At least 5100 but less than 5200
(D) At least 5200 but less than 5300
(E) At least 5300

Solutions to Practice Exam 1

1.	A	
2.	C	
3.	D	
4.	E	
5.	C	
6.	E	
7.	D	
8.	B	
9.	A	
10.	E	
11.	C	
12.	B	
13.	C	
14.	A	
15.	E	
16.	D	
17.	C	
18.	C	
19.	E	
20.	C	

1. A decision maker with utility function $u(w)$ will pay amount C to play a game (random variable) W based on the following relationship: $u(C) = E[u(W)]$.

   For individual X: $E[u_X(W)] = \int_0^{10{,}000} \frac{\sqrt{w}}{10{,}000}\, dw = \frac{200}{3} = u_X(C_X) = \sqrt{C_X}$.

   Solving for C_X results in $C_X = \left(\frac{200}{3}\right)^2 = 4{,}444.44$

   For individual Y: $E[u_Y(W)] = \int_0^{10{,}000} \frac{w}{100} \times \frac{1}{10{,}000}\, dw = 50 = u_Y(C_Y) = \frac{C_Y}{100}$.

   Solving for C_Y results in $C_Y = 5{,}000$

   For individual Z: $E[u_Z(W)] = \int_0^{10{,}000} \left(\frac{w}{100}\right)^2 \times \frac{1}{10{,}000}\, dw = \frac{10{,}000}{3} = u_Z(C_Z) = \left(\frac{C_Z}{100}\right)^2$.

   Solving for C_Z results in $C_Z = 100 \times \sqrt{\frac{10{,}000}{3}} = 5{,}773.50$

   Only Z will pay more than 5,000 for the gamble. In general the concavity of the utility function determines the risk profile of the individual. X has a concave utility function $(u''_X(w) < 0)$ and will be risk-averse and will not be willing to pay more that the expected value (fair cost) of the gamble of 5,000. Y has a linear utility function and will be risk-neutral. Y will be willing to pay at most the expected payoff of the gamble of 5.000 but not more. Z has a convex utility function $(u''_X(w) > 0)$ and will be risk-preferring and will pay more the 5,000 for the gamble.

   Answer A

2. With coinsurance factor α, deductible d, policy limit $\alpha(u-d)$, the amount paid per loss is

   (we are assuming in inflation rate of $r=0$)$Y = \begin{cases} 0 & X \le d \\ \alpha(X-d) & d < X \le u. \\ \alpha(u-d) & X > u \end{cases}$

   In this problem, the coinsurance factor is $\alpha = .8$, the deductible is $d = 5{,}000$, and the policy limit is $.8(u - 5{,}000) = 500{,}000$, so that the maximum covered loss is $u = 630{,}000$.

   The amount paid per loss becomes $Y = \begin{cases} 0 & X \le 5{,}000 \\ 0.80(X-5{,}000) & 5{,}000 < X \le 630{,}000 \\ 500{,}000 & X > 630{,}000 \end{cases}$

   Answer C

3. When a payout occurs, it is 1, 2 or 3 with probability $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{7}{8}$. The number of payouts that are 1,2 or 3 follows a Poisson process with an hourly rate of $5 \times \frac{7}{8} = \frac{35}{8}$.
   The expected number of payouts that are 1, 2 or 3 in 20 minutes, say N, has a Poisson distribution with mean $\frac{35}{8} \times \frac{20}{60} = \frac{35}{24}$. The probability that there are no payouts of 1, 2, or 3 in a given 20 minute period is the probability that $N = 0$, which is $e^{-35/24} = .233$.

   Answer D