ACTEX Learning

Study Manual for Exam P

2nd Edition

Sam A. Broverman, Ph.D., ASA

An SOA Exam

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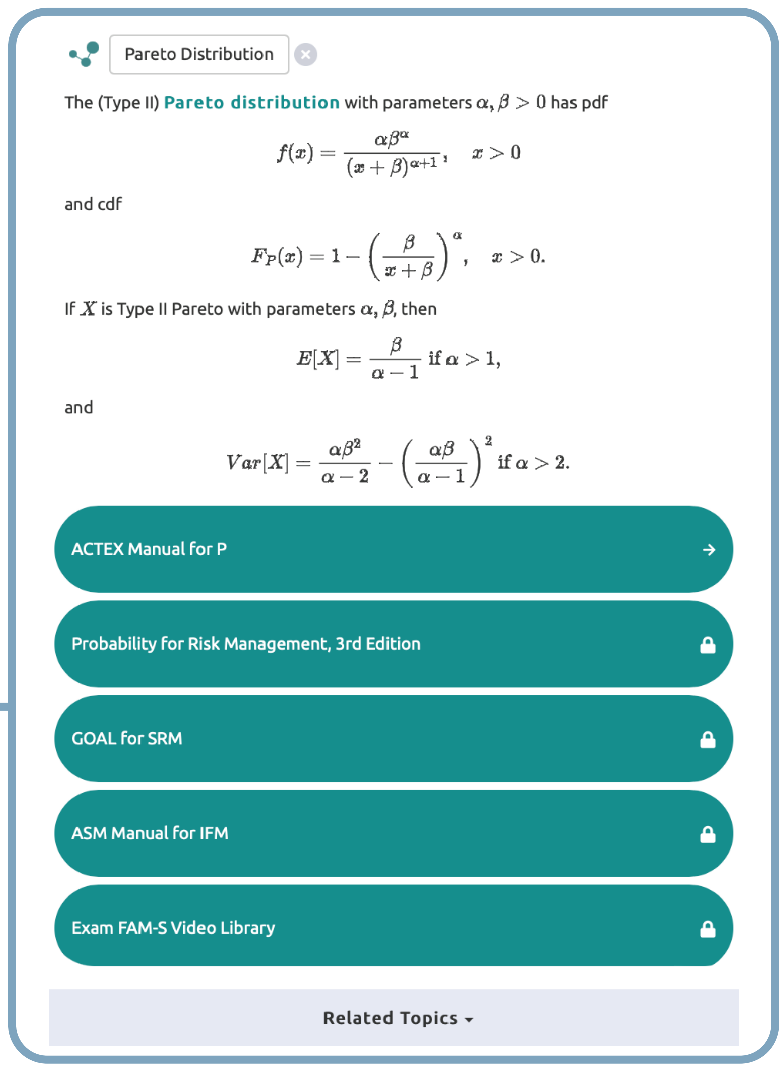

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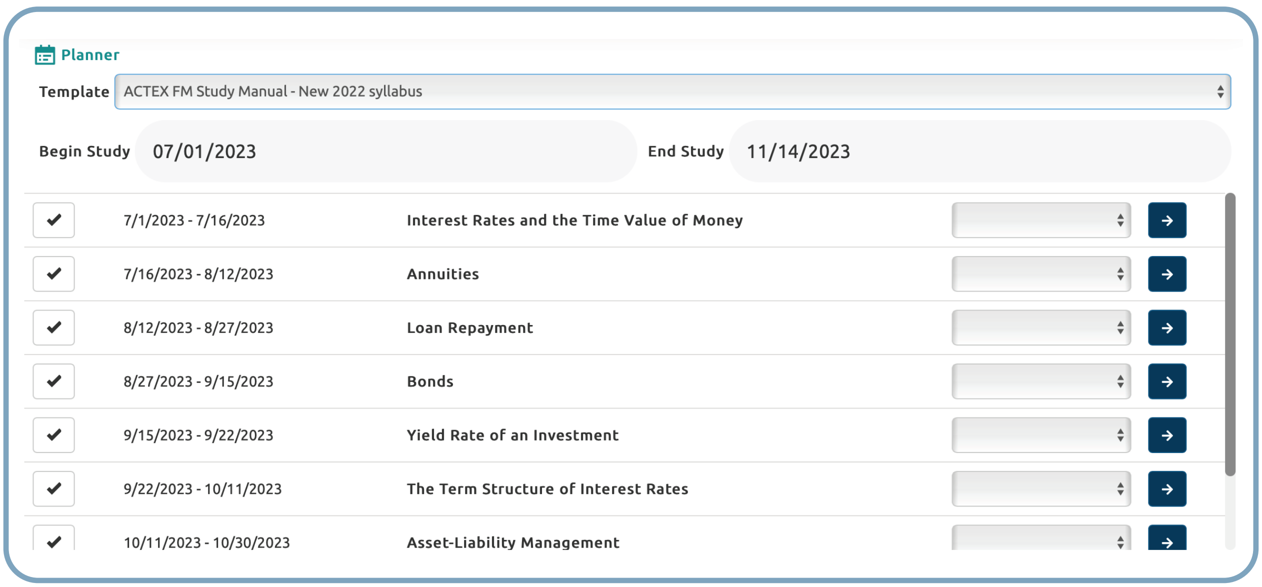

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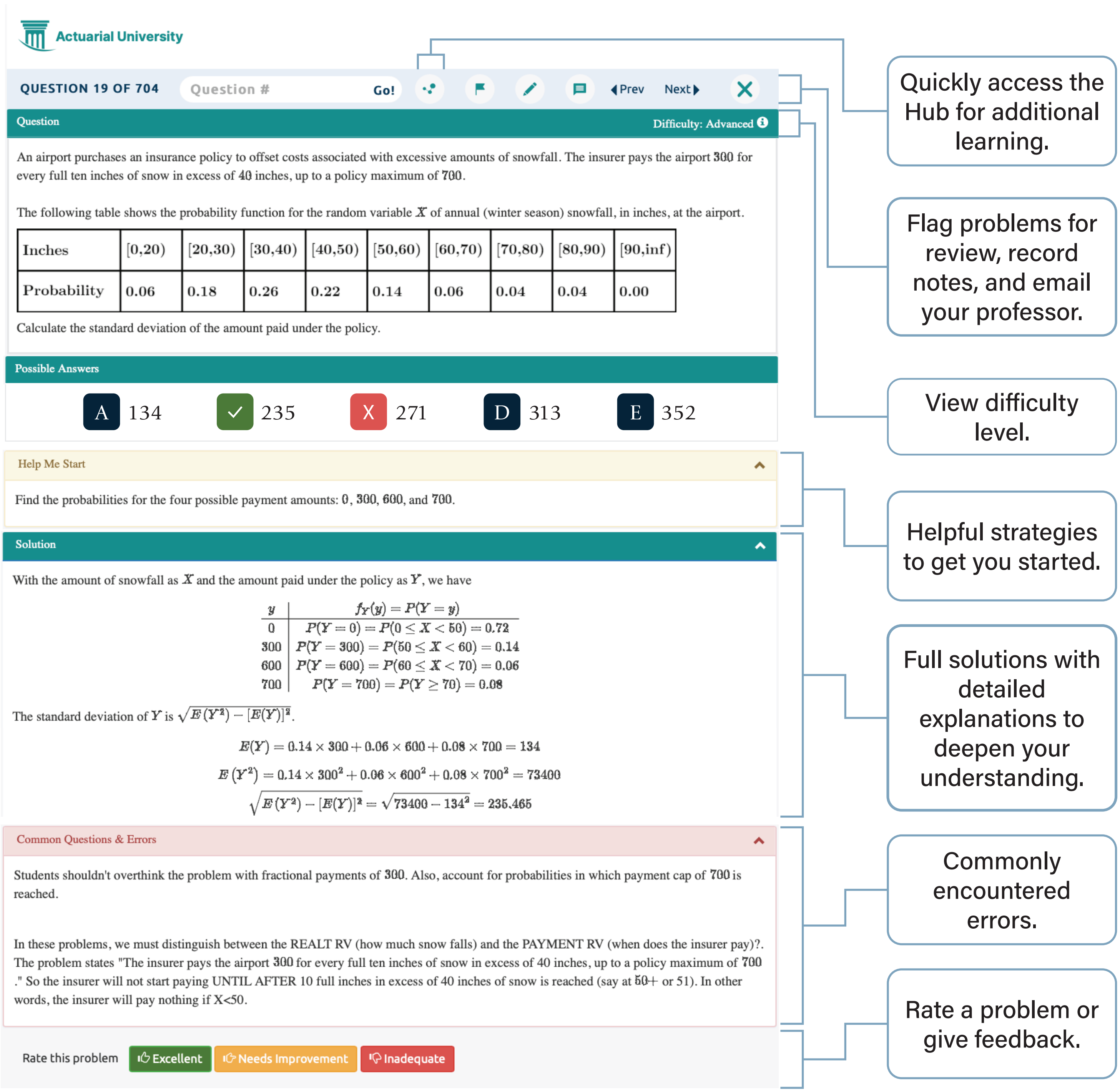

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QUESTION 19 OF 704 | Question # | Go! | Prev | Next

Question — Difficulty: Advanced

An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,inf)
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Possible Answers

A 134   ✓ 235   X 271   D 313   E 352

Help Me Start

Find the probabilities for the four possible payment amounts: 0, 300, 600, and 700.

Solution

With the amount of snowfall as X and the amount paid under the policy as Y, we have

y	$f_Y(y) = P(Y = y)$
0	$P(Y = 0) = P(0 \le X < 50) = 0.72$
300	$P(Y = 300) = P(50 \le X < 60) = 0.14$
600	$P(Y = 600) = P(60 \le X < 70) = 0.06$
700	$P(Y = 700) = P(Y \ge 70) = 0.08$

The standard deviation of Y is $\sqrt{E(Y^2) - [E(Y)]^2}$.

$$E(Y) = 0.14 \times 300 + 0.06 \times 600 + 0.08 \times 700 = 134$$

$$E\left(Y^2\right) = 0.14 \times 300^2 + 0.06 \times 600^2 + 0.08 \times 700^2 = 73400$$

$$\sqrt{E\left(Y^2\right) - [E(Y)]^2} = \sqrt{73400 - 134^2} = 235.465$$

Common Questions & Errors

Students shouldn't overthink the problem with fractional payments of 300. Also, account for probabilities in which payment cap of 700 is reached.

In these problems, we must distinguish between the REALT RV (how much snow falls) and the PAYMENT RV (when does the insurer pay)?. The problem states "The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700 ." So the insurer will not start paying UNTIL AFTER 10 full inches in excess of 40 inches of snow is reached (say at 50+ or 51). In other words, the insurer will pay nothing if X<50.

Rate this problem: Excellent | Needs Improvement | Inadequate

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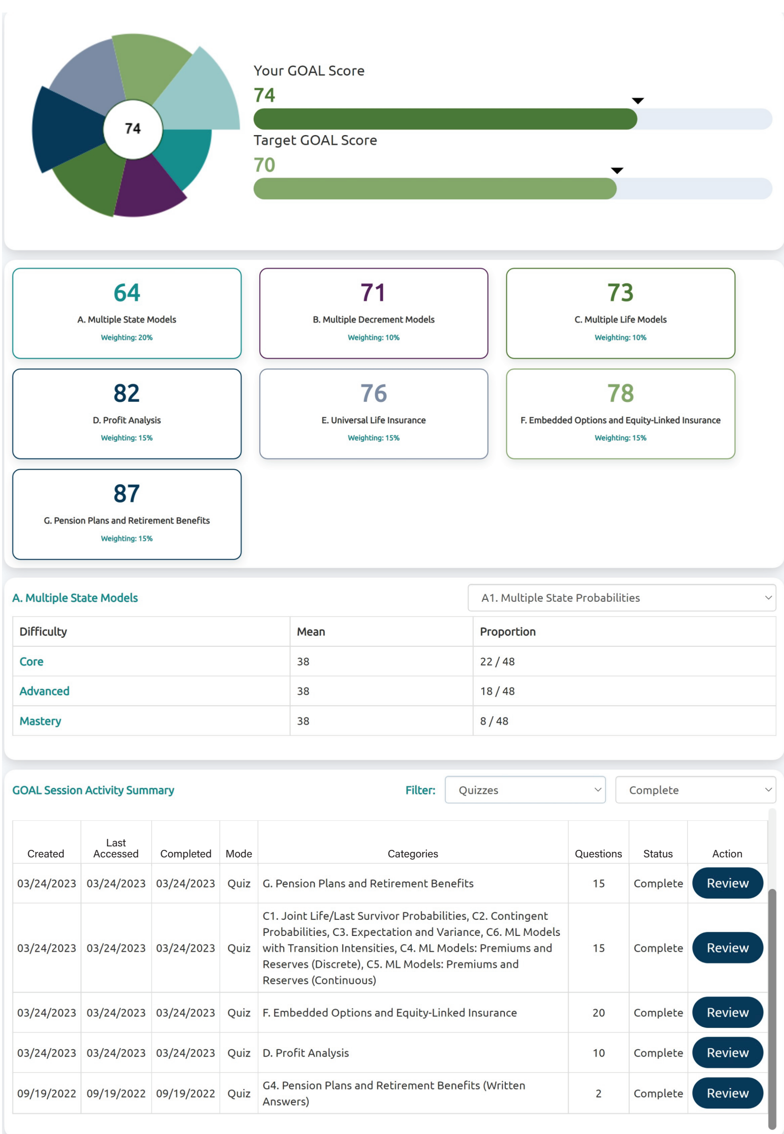

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Contents

Introductory Comments

This study manual is designed to help in the preparation for the Society of Actuaries Exam P. The study manual is divided into two main parts. The first part consists of a summary of notes and illustrative examples related to the material described in the Society of Actuaries (SOA) exam syllabus as well as a series of problem sets and detailed solutions related to each topic. Many of the examples and problems in the problem sets are taken from actual exams (and from the sample question link posted on the SOA website). Note that the symbol "□" will denote the end of an example.

The second part of the study manual consists of thirteen practice exams, with detailed solutions, which are designed to cover the range of material that will be found on the exam. The questions on these practice exams are not from old SOA exams and may be somewhat more challenging, on average, than questions from previous actual exams. Between the section of notes and the section with practice exams I have included the normal distribution table provided with the exam.

I have attempted to be thorough in the coverage of the topics upon which the exam is based. I have been, perhaps, more thorough than necessary on a couple of topics, particularly order statistics in Section 9 of the notes and some risk management topics in Section 10 of the notes. Section 0 of the notes provides a brief review of a few important topics in calculus and algebra. A few mathematical derivations of formulas will be presented but many formulas will be given without derivation.

This manual will be most effective for those who have had courses in college calculus at least to the sophomore level and courses in probability to the sophomore or junior level.

If you are taking Exam P for the first time, be aware that a most crucial aspect of the exam is the limited time given to take the exam (3 hours). Time management is extremely important, as is being able to work very quickly and accurately. Continual drill on important concepts and formulas by working through many problems will be helpful. It is also very important to be disciplined enough while taking the exam to avoid spending an inordinate amount of time on any one question. If the formulas and reasoning that will be needed to solve a particular question are not clear within 2 or 3 minutes of starting the question, the question should be abandoned (and returned to later if time permits). Using the exams in the second part of this study manual and simulating exam conditions will also help give you a feeling for the actual exam experience.

If you have any comments, criticisms or compliments regarding this study manual, please contact the publisher, ACTEX, or you may contact me directly at the e-mail address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you bring them to my attention. Any errors that are found will be posted in an errata file at the ACTEX website, www.actexlearning.com.

It is my sincere hope that you find this study manual helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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Section 2

Conditional Probability and Independence

2.1 Definition of Conditional Probability

Conditional probability of event B given event A:

If $P[A] > 0$, then $\boldsymbol{P[B|A] = \frac{P[B \cap A]}{P[A]}}$ is defined to be the conditional probability **that event** B **occurs given that event** A **has occurred**. Events A and B may be related so that if we know that event A has occurred, the **conditional probability of event B occurring given that event A has occurred** might not be the same as the unconditional probability of event B occurring if we had no knowledge about the occurrence of event A. For instance, if a fair 6-sided die is tossed and if we know that the outcome is even, then the conditional probability is 0 of tossing a 3 given that the toss is even. If we did not know that the toss was even, if we had no knowledge of the nature of the toss, then tossing a 3 would have an unconditional probability of $\frac{1}{6}$, the same as all other possible tosses that could occur.

When we condition on event A, we are assuming that event A has occurred so that A becomes the new probability space, and all conditional events must take place within event A (the new probability space). Dividing by $P[A]$ scales all probabilities so that A is the entire probability space, and $P[A|A] = 1$. To say that event B has occurred given that event A has occurred means that both B and A (i.e., $B \cap A$) have occurred within the probability space A. This explains the numerator $P[B \cap A]$ in the definition of the conditional probability $P[B|A]$.

The expression $P[B|A] = \frac{P[B \cap A]}{P[A]}$ can be rewritten in the equivalent form

$$\boldsymbol{P[B \cap A] = P[B|A] \times P[A]}.$$

This is referred to as the **multiplication rule**.

Conditional probabilities satisfy all of the basic relationships satisfied by probabilities outlined in the previous section of this study manual. For instance $P[B'|A] = 1 - P[B|A]$ and $P[B \cup C|A] = P[B|A] + P[C|A] - P[B \cap C|A]$.

Example 2.1.
Suppose that a fair six-sided die is tossed. The probability space is $S = \{1,2,3,4,5,6\}$. We define the following events:
A =“the number tossed is even”$= \{2,4,6\}$,
B =“the number tossed is ≤ 3”$= \{1,2,3\}$,
C =“the number tossed is a 1 or a 2”$= \{1,2\}$,
D =“the number tossed doesn’t start with the letters ‘f’ or ’t’ ”$= \{1,6\}$.

The conditional probability of B given A is

$$P[B|A] = \frac{P[B\cap A]}{P[A]} = \frac{P[\{1,2,3\}\cap\{2,4,6\}]}{P[\{2,4,6\}]} = \frac{P[\{2\}]}{P[\{2,4,6\}]} = \frac{1/6}{1/2} = \frac{1}{3}.$$

We can interpret this conditional probability as follows. Knowing that event A has occurred means that the toss must be 2, 4 or 6. Since the original 6 possible tosses of a die were equally likely, if we are given the additional information that the toss is 2, 4 or 6, it seems reasonable that each of those possible tosses is equally likely, each with a probability of $\frac{1}{3}$. Then within the reduced probability space A, the (conditional) probability that event B occurs is the probability, in the reduced space, of tossing a 2 out of the equally likely possibilities of tossing a 2, 4 or 6; this is $\frac{1}{3}$.

For events B and C, the conditional probability of B given C is $P[B|C] = \frac{P[B\cap C]}{P[C]}$. Since $B\cap C = C$ this conditional probability is $\frac{P[C]}{P[C]} = 1$. Another way of looking at this is to note that C is a subevent of B (i.e. $C \subset B$ so that $C\cap B = C$), and to say that C has occurred means that the toss is 1 or 2. It is then guaranteed that event B has occurred (the toss is a 1, 2 or 3) given that C has occurred.

The conditional probability of A given C is $P[A|C] = \frac{P[2,4,6\cap 1,2]}{P[1,2]} = \frac{P[2]}{P[1,2]} = \frac{1}{2}$. □

Example 2.2.
If $P[A] = \frac{1}{6}$ and $P[B] = \frac{5}{12}$, and $P[A|B] + P[B|A] = \frac{7}{10}$, find $P[A\cap B]$.

Solution:
$P[B|A] = \frac{P[A\cap B]}{P[A]} = 6\times P[A\cap B]$ and $P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{12}{5}\times P[A\cap B]$
$\rightarrow\ (6+\frac{12}{5})\times P[A\cap B] = \frac{7}{10}\ \rightarrow\ P[A\cap B] = \frac{1}{12}$. □

NOTE:
The following important manipulation of event probabilities arises from time to time:

$$\boldsymbol{P[B] = P[B|A]\times P[A] + P[B|A']\times P[A']}.$$

The relationship in the note above is a version of the **Law of Total Probability**. This relationship is valid since for any events A and B, we have $P[B] = P[B\cap A] + P[B\cap A']$. We then use the relationships $P[B\cap A] = P[B|A]\times P[A]$ and $P[B\cap A'] = P[B|A']\times P[A']$. If we know the conditional probabilities for event B given some other event A (i.e., $P[B|A]$), and if we also know the conditional probability of B given the complement A' (i.e., $P[B|A']$), and if we are given the (unconditional) probability of event A (i.e., $P[A]$ from which we know $P[A'] = 1-P[A]$), then we have all the components we need to find find the (unconditional) probability of event B using the equation in the note above. An application of this concept occurs when an experiment has two (or more) steps. The following example illustrates this idea.

Example 2.3.
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. An Urn is chosen at random, and a ball is randomly selected from that Urn. Find the probability that the ball chosen is white.

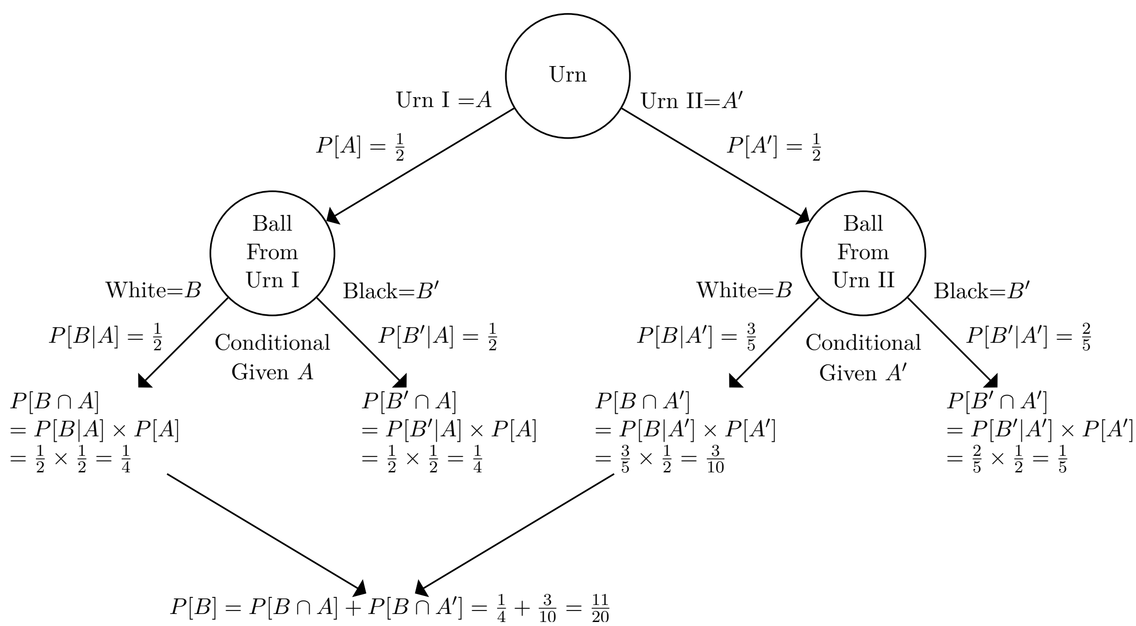
Solution:

Let A be the event that Urn I is chosen and A' is the event that Urn II is chosen. The implicit assumption is that both Urns are equally likely to be chosen (this is the meaning of "an Urn is chosen at random"). Therefore, $P[A] = \frac{1}{2}$ and $P[A'] = \frac{1}{2}$. Let B be the event that the ball chosen is white. If we know that Urn I was chosen, then there is $\frac{1}{2}$ probability of choosing a white ball (2 white out of 4 balls, it is assumed that each ball has the same chance of being chosen); this can be described as $P[B|A] = \frac{1}{2}$. In a similar way, if Urn II is chosen, then $P[B|A'] = \frac{3}{5}$ (3 white out of 5 balls). We can now apply the relationship described prior to this example. $P[B \cap A] = P[B|A] \times P[A] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and $P[B \cap A'] = P[B|A'] \times P[A'] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.
And then we get $P[B] = P[B \cap A] + P[B \cap A'] = \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$.

The order of calculations (1-2-3 labeled below) can be summarized in the following table:

	A	A'
B	**1.**$P[B \cap A] = P[B|A] \times P[A]$	**2.**$P[B \cap A'] = P[B|A'] \times P[A']$
	3.$P[B] = P[B \cap A] + P[B \cap A]$	

An event tree diagram, shown below, is another way of illustrating the probability relationships.



□

NOTE:

An exam question may state that "an item is to be **chosen at random** from a collection of items". Unless there is an indication otherwise, this is interpreted to mean that **each item has the same chance of being chosen**. Also, if we are told that a **fair coin is tossed randomly**, then we interpret that to mean that **the head and tail each have the probability of 0.5 occurring**. Of course, if we are told that the coin is "loaded" so that the probability of tossing a head is $\frac{2}{3}$ and tail is $\frac{1}{3}$, then random toss means the head and tail will occur with those stated probabilities.

2.2 Bayes' Rule, Bayes' Theorem and the Law of Total Probability

Bayes' rule and Bayes' Theorem (basic form):

For any events A and B with $P[B] > 0$,

$$\boldsymbol{P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B|A] \times P[A]}{P[B|A] \times P[A] + P[B|A'] \times P[A']}}.$$

The usual way that this is applied is in the case in which we are given the values of

$P[A]$, $P[B|A]$ and $P[B|A']$ (from $P[A]$ we get $P[A'] = 1 - P[A]$),

and we are asked to find $P[A|B]$ (in other words, we are asked to "turn around" the conditioning of the events A and B). We can summarize this process by calculating the quantities in the following "table of calculations" in the order indicated numerically (1-2-3-4) (other entries in the table are not necessary in this calculation, but might be needed in related calculations).

	A, $P[A]$ given	A', $P[A']$ given
B	$P[B\|A]$ given **1 :** $P[B \cap A] = P[B\|A] \times P[A]$	$P[B\|A']$ given **2 :** $P[B \cap A'] = P[B\|A'] \times P[A']$

$\Downarrow$ $\Downarrow$

3 : $P[B] = P[B \cap A] + P[B \cap A']$

Step **4**: $P[A|B] = \frac{P[A \cap B]}{P[B]}$

Also, we can find

B'	$P[B'\|A] = 1 - P[B\|A]$ $P[B' \cap A] = P[B'\|A] \times P[A]$	$P[B'\|A'] = 1 - P[B\|A']$ $P[B' \cap A'] = P[B'\|A'] \times P[A']$

and $P[B'] = P[B' \cap A] + P[B' \cap A']$ (but we could have found $P[B']$ from $P[B'] = 1 - P[B]$, once $P[B]$ was found).

This can also be summarized in the following sequence of calculations.

$$P[A], P[B|A], \text{ given} \qquad P[A'] = 1 - P[A], P[B|A'], \text{ given}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$P[B \cap A] = P[B|A] \times P[A] \qquad P[B \cap A'] = P[B|A'] \times P[A']$$

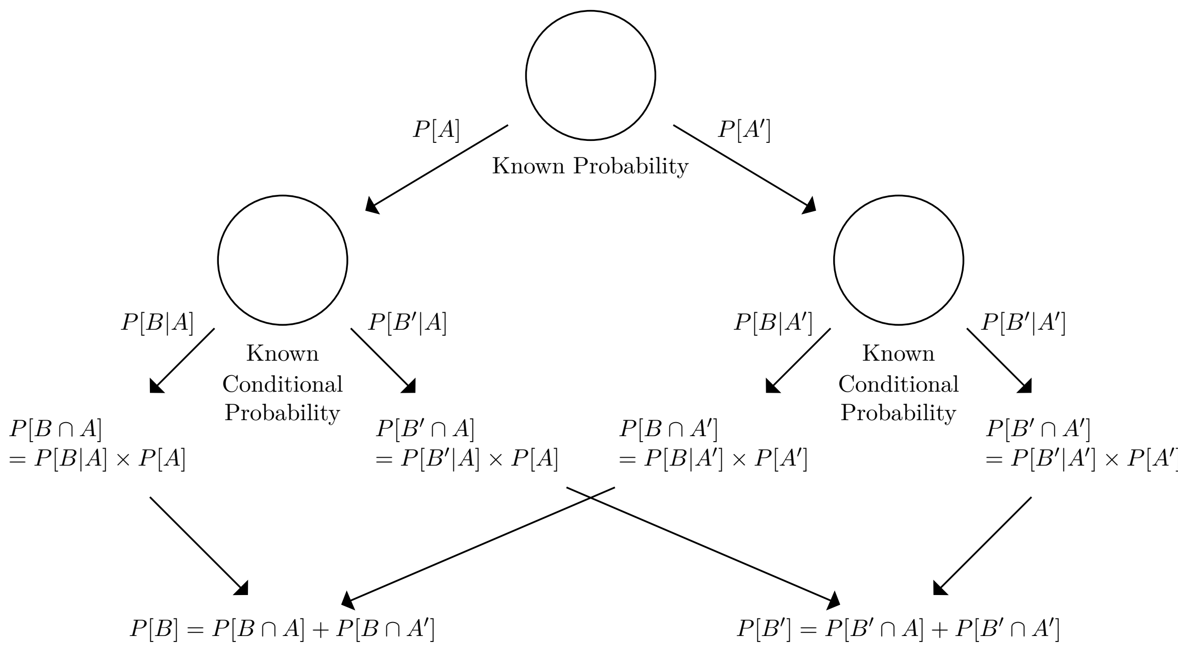
$$P[B] = P[B \cap A] + P[B \cap A']$$

Algebraically, we have done the following calculation:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B]}{P[B \cap A] + P[B \cap A']} = \frac{P[B|A] \times P[A]}{P[B|A] \times P[A] + P[B|A'] \times P[A']},$$

where all the factors on the right side of the equation were originally known. Note that the numerator is one of the components of the denominator. The following event tree is similar to the one in Example 2.3, illustrating the probability relationships.

Note that at the bottom of the event tree, $P(B')$ is also equal to $1 - P(B)$.



NOTE:

Exam questions that involve conditional probability and make use of Bayes rule (and its extended form reviewed below) occur frequently. The key starting point is identifying and labeling unconditional events and conditional events and their probabilities in an efficient way.

Example 2.4.
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. One ball is chosen at random from Urn I and transferred to Urn II, and then a ball is chosen at random from Urn II. The ball chosen from Urn II is observed to be white. Find the probability that the ball transferred from Urn I to Urn II was white.
Solution:

Let A denote the event that the ball transferred from Urn I to Urn II was white and let B denote the event that the ball chosen from Urn II is white. We are asked to find $P[A|B]$.

From the simple nature of the situation (and the usual default assumption of uniformity in such a situation, meaning that all balls are equally likely to be chosen from Urn I in the first step), we have $P[A] = \frac{1}{2}$ (2 of the 4 balls in Urn I are white), and $P[A'] = \frac{1}{2}$. These are the unconditional probabilities of events A and A', the probability of choosing a white or a black ball from Urn I without any further information. Bayes' Rule will allow us to update the probability to a conditional probability of the color of the first ball chosen from Urn I using the additional given information about the color of ball that is subsequently chosen from Urn II after the first ball chosen is added to Urn II.

If the ball transferred is white, then Urn II has 4 white and 2 black balls, and the probability of choosing a white ball out of Urn II is $\frac{2}{3}$; this is $P[B|A] = \frac{2}{3}$.
If the ball transferred is black, then Urn II has 3 white and 3 black balls, and the probability of choosing a white ball out of Urn II is $\frac{1}{2}$; this is $P[B|A'] = \frac{1}{2}$.

All of the information needed has been identified. As in the "table of calculation" described a little earlier, we do the calculations in the following order:

1. $P[B \cap A] = P[B|A] \times P[A] = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$
2. $P[B \cap A'] = P[B|A'] \times P[A'] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
3. $P[B] = P[B \cap A] + P[B \cap A'] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
4. $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{1/3}{7/12} = \frac{4}{7}$ □

Example 2.5.
Identical twins come from the same egg and hence are of the same sex. Fraternal twins have a 50-50 chance of being the same sex. Among twins, the probability of a fraternal set is p and an identical set is $q = 1 - p$. If the next set of twins are of the same sex, formulate the probability that they are identical in terms of q.

Solution:

Let B be the event "the next set of twins are of the same sex", and let A be the event "the next sets of twins are identical". We are given $P[B|A] = 1$, $P[B|A'] = 0.5$, $P[A] = q$, and $P[A'] = p = 1 - q$.

Then $P[A|B] = \frac{P[B \cap A]}{P[B]}$ is the probability we are asked to find.

But $P[B \cap A] = P[B|A] \times P[A] = q$, and $P[B \cap A'] = P[B|A'] \times P[A'] = 0.5p$.

Thus, $P[B] = P[B \cap A] + P[B \cap A'] = q + 0.5 \times p = q + 0.5 \times (1 - q) = 0.5 \times (1 + q)$, and $P[A|B] = \frac{q}{0.5 \times (1+q)}$. This can be summarized in the following table

	A = identical, $P[A] = q$	A' = not identical, $P[A'] = 1 - q$
B = same sex	$P[B\|A] = 1$ (given) , $P[B \cap A]$ $= P[B\|A] \times P[A] = q$	$P[B\|A'] = 0.5$ (given) , $P[B \cap A']$ $= P[B\|A'] \times P[A'] = 0.5 \times (1 - q)$

$$\Downarrow$$

$$P[B] = P[B \cap A] + P[B \cap A'] = q + 0.5 \times (1 - q) = 0.5 \times (1 + q)$$

Then, $P[A|B] = \frac{P[B \cap A]}{P[B]} = \frac{q}{0.5 \times (1+q)}$.

The event tree shown earlier can be applied to this example. □

Bayes' rule and Bayes' Theorem (extended form):
Suppose $A_1, A_2, \ldots, A_n$ form a partition of the entire probability space S and B is any event.
The **General Law of Total Probability** is $P[B] = \sum_{i=1}^{n} P[B \cap A_i] = \sum_{i=1}^{n} P[B|A_i] \times P[A_i]$.
The **General Form of Bayes Theorem** is

$$P[A_j|B] = \frac{P[B \cap A_j]}{P[B]} = \frac{P[B \cap A_j]}{\sum_{i=1}^{n} P[B \cap A_i]} = \frac{P[B|A_j] \times P[A_j]}{\sum_{i=1}^{n} P[B|A_i] \times P[A_i]} \text{ for each } j = 1,2,\ldots,n.$$

If $n = 3$, and events A_1, A_2, and A_3 form a partition of the full probability space, then

$$\begin{aligned} P[A_1|B] &= \frac{P[A_1 \cap B]}{P[B]} = \frac{P[B|A_1] \times P[A_1]}{P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3]} \\ &= \frac{P[B|A_1] \times P[A_1]}{P[B|A_1] \times P[A_1] + P[B|A_2] \times P[A_2] + P[B|A_3] \times P[A_3]}. \end{aligned}$$

Similar relationships hold for $P[A_2|B]$ and $P[A_3|B]$. The relationship in the denominator, $P[B] = \sum_{i=1}^{n} P[B|A_i] \times P[A_i]$ is the general Law of Total Probability. The most basic form of Bayes' rule is the case in which the partition consists of just two events, A and A'. This was presented a little earlier.

The main application of Bayes' rule occurs in the situation in which the $P[A_i]$ probabilities are known and the $P[B|A_h]$ probabilities are known, and we are asked to find $P[A_j|B]$ for one or more of the j's. The series of calculations can be summarized in a table similar to that for the basic form of Bayes' rule. This is illustrated in the following example.

Example 2.6.
Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

Solution:

The event "a 6 is thrown" is denoted by B and A_1,A_2 and A_3 denote the events that die 1, die 2 and die 3 was chosen. $P[A_1|B] = \dfrac{P[A_1 \cap B]}{P[B]} = \dfrac{P[B|A_1] \times P[A_1]}{P[B]} = \dfrac{p \times \frac{1}{3}}{P[B]}$. But

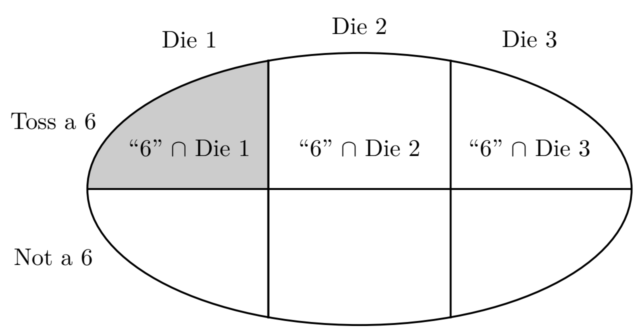
$$\begin{aligned}
P[B] &= P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3] \\
&= P[B|A_1] \times P[A_1] + P[B|A_2] \times P[A_2] + P[B|A_3] \times P[A_3] \\
&= p \times \frac{1}{3} + q \times \frac{1}{3} + r \times \frac{1}{3} = \frac{p+q+r}{3} \\
&\Rightarrow \ P[A_1|B] = \frac{p \times \frac{1}{3}}{P[B]} = \frac{p \times \frac{1}{3}}{(p+q+r) \times \frac{1}{3}} = \frac{p}{p+q+r}.
\end{aligned}$$

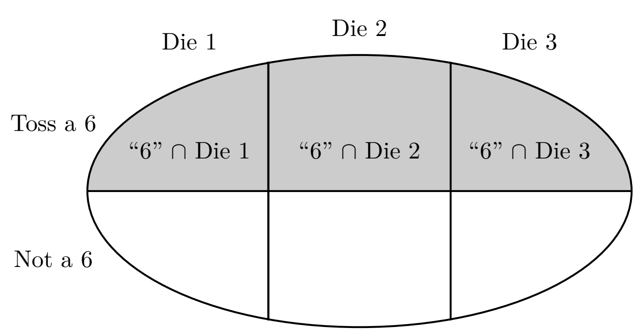
These calculations can be summarized in the following table:

	Die 1, $P[A_1] = \frac{1}{3}$ (given)	Die 2, $P[A_2] = \frac{1}{3}$ (given)	Die 3, $P[A_3] = \frac{1}{3}$ (given)
Toss "6" B	$P[B|A_1] = p$ (given) Then $P[B \cap A_1]$ $= P[B|A_1] \times P[A_1] = p \times \frac{1}{3}$	$P[B|A_2] = q$ (given) Then $P[B \cap A_2]$ $= P[B|A_2] \times P[A_2] = q \times \frac{1}{3}$	$P[B|A_3] = r$ (given) Then $P[B \cap A_3]$ $= P[B|A_3] \times P[A_3] = r \times \frac{1}{3}$

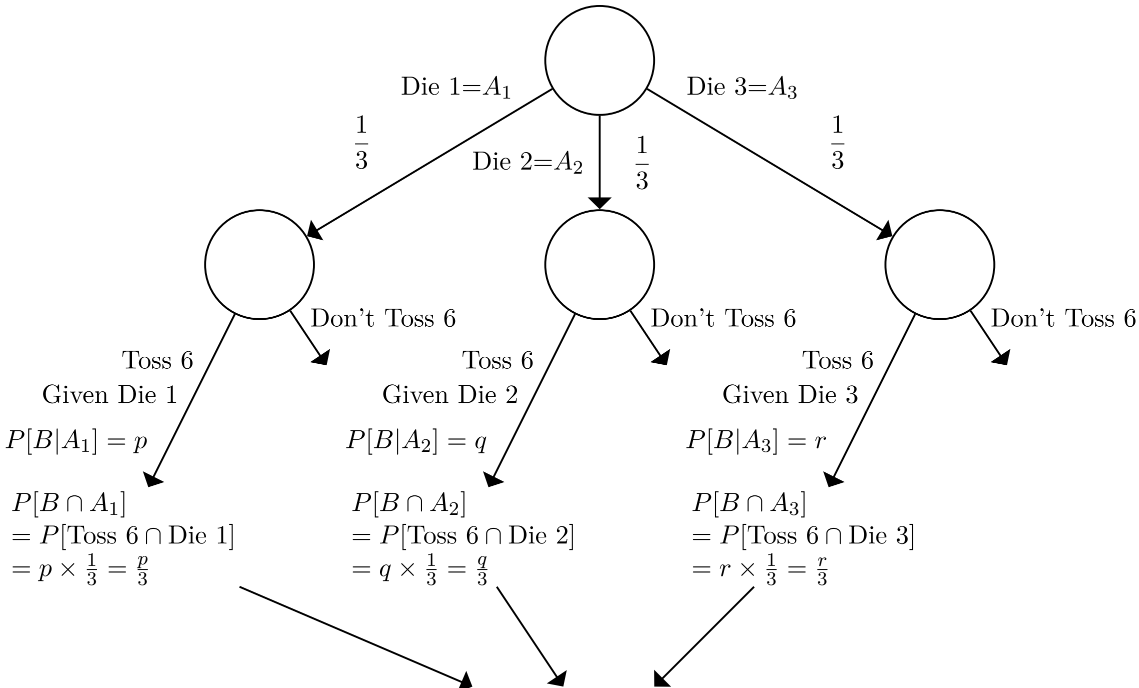
$P[B] = p \times \frac{1}{3} + q \times \frac{1}{3} + r \times \frac{1}{3} = \frac{1}{3} \times (p+q+r)$.

In terms of Venn diagrams, the conditional probability is the ratio of the shaded area probability in the first diagram to the shaded area probability in the second diagram.





The event tree representing the probabilities has three branches at the top node to represent the three die types that can be chosen in the first step of the process.



$$\begin{aligned} P[\text{Toss } 6] &= P[B \cap A_1] + P[B \cap A_2] + P[B \cap A_3] \\ &= P[\text{Toss } 6 \cap \text{Die } 1] + P[\text{Toss } 6 \cap \text{Die } 2] + P[\text{Toss } 6 \cap \text{Die } 3] \\ &= \frac{p}{3} + \frac{q}{3} + \frac{r}{3} = \frac{p+q+r}{3} \end{aligned}$$

□

In Example 2.6 there is some symmetry to the situation and general reasoning can provide a shortened solution. In the conditional probability

$$P[\text{die } 1|\text{“6”}] = \frac{P[(\text{die } 1) \cap (\text{“6”})]}{P[\text{“6”}]},$$

we can think of the denominator as the combination of the three possible ways a "6" can occur from the three dice combined, $p+q+r$, and we can think of the numerator as the "6" occurring from die 1, with probability p. Then the conditional probability is the fraction $\frac{p}{p+q+r}$. The symmetry involved here is in the assumption that each die was equally likely to be chosen, so there is a $\frac{1}{3}$ chance of any one die being chosen. This factor of $\frac{1}{3}$ cancels in the numerator and denominator of $\frac{p \times \frac{1}{3}}{(p+q+r) \times \frac{1}{3}}$. If we had not had this symmetry, we would have to apply different "weights" to the three dice.

Another example of using this sort of symmetry to simplify probability calculations is a variation on Example 2.3 above. Suppose that Urn I has 2 white and 3 black balls and Urn II has 4 white and 1 black balls. An Urn is chosen at random and a ball is chosen. The reader should verify using the usual conditional probability rules that the probability of choosing a white ball is $\frac{6}{10}$. This can also be seen by noting that if we consider that of the 10 balls that are in the two Urns combined, 6 are white, so that the chance of picking a white ball out of the 10 is $\frac{6}{10}$. Suppose that we do not know which Urn was chosen but we are told that the ball chosen is white and we wish to apply Bayes' Rule to find the conditional probability that Urn I was chosen given that the ball color is white. The reader should use the usual conditional probability rules to verify that this conditional probability is $\frac{0.4}{1.2} = \frac{1}{3}$.

The similarity of the modification to Example 2.3 to the situation in Example 2.6 is that in the modification to Example 2.3 just presented, both Urns have the same initial chance of $\frac{1}{2}$ of being chosen and in Example 2.6, each of the three dice has the same initial chance of $\frac{1}{3}$ of being chosen. We can see this idea in the general framework of Bayes' Rule. Suppose each of the partition events $A_1, A_2, ..., A_n$ has the same probability of $\frac{1}{n}$. Then

$$P[A_j|B] = \frac{P[B|A_j] \times P[A_j]}{\sum_{i=1}^{n} P[B|A_i] \times P[A_i]} = \frac{P[B|A_j] \times \frac{1}{n}}{\sum_{i=1}^{n} P[B|A_i] \times \frac{1}{n}} = \frac{P[B|A_j]}{\sum_{i=1}^{n} P[B|A_i]} \text{ for each } j = 1,2,...,n.$$

In the modification to Example 2.3, $n = 2$, A_1 and A_2 are "choosing Urns I and II", respectively and B is "the ball is white". Then

$$P[\text{Urn I}|\text{white}] = P[A_1|B] = \frac{P[B|A_1]}{P[B|A_1] + P[B|A_2]} = \frac{2/5}{2/5 + 4/5} = \frac{0.4}{1.2} = \frac{1}{3}$$

In Example 2.6, $n = 3$ and A_i is "choose die i" and B is "a 6 is tossed", so that

$$P[\text{Die 1}|\text{"6"}] = P[A_1|B] = \frac{P[B|A_1]}{P[B|A_1] + P[B|A_2] + P[B|A_3]} = \frac{p}{p+q+r}$$

In the modification to Example 2.3 there is additional symmetry that can be used to simplify the determination of the conditional probability. We see that knowing that the ball is white and knowing that 2 white balls are in Urn I out of a total of $2 + 4 = 6$ white balls in the two Urns combined, and knowing that both Urns have the same number of balls (5 balls in each Urn), the conditional probability of having chosen Urn I given that the ball chosen is white is $\frac{2}{6}$. Again, this is only valid in this case because the initial probabilities of choosing Urn I or II are both $\frac{1}{2}$ and both Urns have the same number of balls.

2.3 Independent Events

Independent events A and B:
If events A and B satisfy the relationship $P[A \cap B] = P[A] \times P[B]$, then the events are said to be **independent** or **stochastically independent** or **statistically independent**. The independence of (non-empty) events A and B is equivalent to $P[A|B] = P[A]$, and also is equivalent to $P[B|A] = P[B]$.

Example 2.1. (continued): A fair six-sided die is tossed.
A = "the number tossed is even" = {2,4,6},
B = "the number tossed is ≤ 3" = {1,2,3},
C = "the number tossed is a 1 or a 2" = {1,2},
D = "the number tossed doesn't start with the letters 'f' or 't'" = {1,6}.

We have the following probabilities: $P[A] = \frac{1}{2}, P[B] = \frac{1}{2}, P[C] = \frac{1}{3}, P[D] = \frac{1}{3}$.

A and B are not independent since $\frac{1}{6} = P[\{2\}] = P[A \cap B] \neq P[A] \times P[B] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. We can also see that A and B are not independent because $P[B|A] = \frac{1}{3} \neq \frac{1}{2} = P[B]$.

Also, B and C are not independent, since $P[B \cap C] = \frac{1}{3} \neq \frac{1}{2} \times \frac{1}{3} = P[B] \times P[C]$ (also since $P[B|C] = 1 \neq \frac{1}{2} = P[B]$).

Events A and C are independent, since $P[A \cap C] = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = P[A] \times P[C]$ (alternatively, $P[A|C] = \frac{1}{2} = P[A]$, also indicating that A and C are independent).
The reader should check that both A and B are independent of D. □

Mutually independent events: Events $A_1, A_2, \ldots, A_n$ are said to be **mutually independent** if the following relationships are satisfied. For any two events, say A_i and A_j, we have $P[A_i \cap A_j] = P[A_i] \times P[A_j]$. For any three events, Say A_i, A_j, A_k, we have $P[A_i \cap A_j \cap A_k] = P[A_i] \times P[A_j] \times P[A_k]$. This must be true for any four events, or five events, etc.

Some rules concerning conditional probability and independence are:

(i) $P[A \cap B] = P[B|A] \times P[A] = P[A|B] \times P[B]$ for any events A and B

(ii) If $P[A_1 \cap A_2 \cap \cdots \cap A_{n-1}] > 0$, then

$$P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1] \times P[A_2|A_1] \times P[A_3|A_1 \cap A_2] \times \cdots \times P[A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}]$$

(iii) $\boldsymbol{P[A'|B] = 1 - P[A|B]}$

(iv) $P[A \cup B|C] = P[A|C] + P[B|C] - P[A \cap B|C]$

(v) If $A \subset B$ then $P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]}{P[B]}$, and $P[B|A] = 1$;
properties (iv) and (v) are the same properties satisfied by unconditional events

(vi) If A and B are independent events then A' and B are independent events, A and B' are independent events, and A' and B' are independent events

(vii) Since $P[\emptyset] = P[\emptyset \cap A] = 0 = P[\emptyset] \times P[A]$ for any event A, it follows that $\emptyset$ is independent of any event A

Example 2.7.
Suppose that events A and B are independent. Find the probability, in terms of $P[A]$ and $P[B]$, that exactly one of the events A and B occurs.

Solution:

$P[\text{exactly one of } A \text{ and } B] = P[(A \cap B') \cup (A' \cap B)]$.
Since $A \cap B'$ and $A' \cap B$ are mutually exclusive, it follows that
$P[\text{exactly one of } A \text{ and } B] = P[A \cap B'] + P[A' \cap B]$. Since A and B are independent, it follows that A and B' are also independent, as are B and A'.
Then $P[(A \cap B') \cup (A' \cap B)] = P[A] \times P[B'] + P[B] \times P[A']$
$= P[A] \times (1 - P[B]) + P[B] \times (1 - P[A]) = P[A] + P[B] - 2P[A] \times P[B]$ □

Example 2.8.

In a survey of 94 students, the following data was obtained.

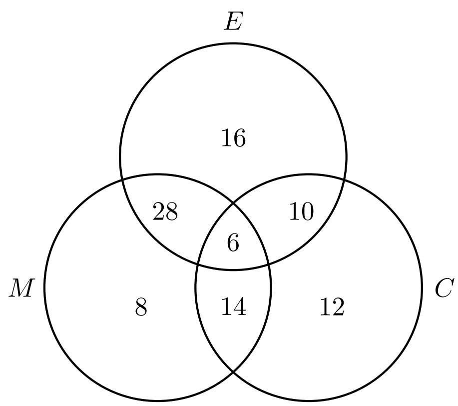
60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects. Find the following proportions.

(i) Of those who took Math, the proportion who took neither English nor Chemistry,

(ii) Of those who took English or Math, the proportion who also took Chemistry.

Solution:

The following diagram illustrates how the numbers of students can be deconstructed. The situation in this example is very similar to that of Example 1.5 earlier in this study manual and the diagram below is found in the same way as it was in Example 1.5. We calculate probability as proportion of the numbers in the various subsets.



(i) We are given that 56 students took Math, and from the diagram we see that 8 of them took neither English nor Chemistry.

$$P[E' \cap C'|M] = \frac{P[E' \cap C' \cap M]}{P[M]} = \frac{n(E' \cap C' \cap M)}{n(M)} = \frac{8}{56} = \frac{1}{7}.$$

(ii) The number of students who took English or Math or both is $n(E \cup M) = n(E) + n(M) - n(E \cap M) = 60 + 56 - 34 = 82$. Alternatively, from the diagram we see that this number is $8 + 14 + 6 + 28 + 16 + 10 = 82$. It can also be seen from the diagram that the number of those who took English or Math and also took Chemistry is $14 + 6 + 10 = 30 = n[(E \cup M) \cap C]$. Then

$$P[C|E \cup M] = \frac{P[(E \cup M) \cap C]}{P[E \cup M]} = \frac{n((E \cup M) \cap C)}{n(E \cup M)} = \frac{30}{82} = \frac{15}{41}. \qquad \square$$

Example 2.9.
A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have irons (I), 55% have electric toasters (T), 50% have $I \cap R$, 40% have $R \cap T$, 30% have $I \cap T$, and 20% have all three $R \cap I \cap T$.

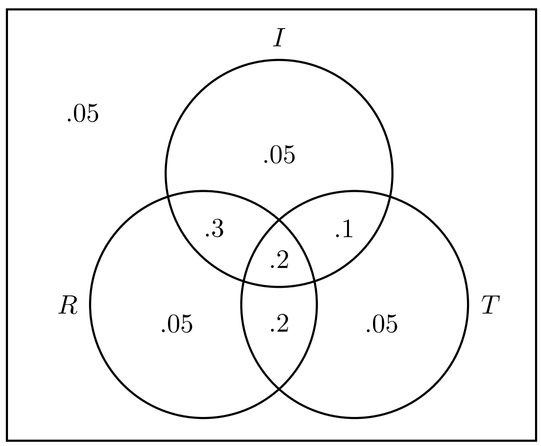
Find the following proportions:

(i) Of those households that have a toaster, find the proportion that also have a radio.

(ii) Of those households that have a toaster but no iron, find the proportion that have a radio.

Solution:

This is a continuation of Example 1.4 given earlier in the study manual. Again, proportion is regarded as the same as probability. The diagram below deconstructs the three events.



(i) This is $P[R|T]$. The language "of those households that have a toaster" means, "given that the household has a toaster", so we are being asked for a conditional probability. Then, $P[R|T] = \frac{P[R \cap T]}{P[T]} = \frac{0.4}{0.55} = \frac{8}{11}$.

(ii) This is $P[R|T \cap I'] = \frac{P[R \cap T \cap I']}{P[T \cap I']} = \frac{.2}{.25} = \frac{4}{5}$. □

Example 2.8 presented a "population" of 94 individuals, each with some combination of various properties (took English, took Math, took Chemistry). We found conditional probabilities involving the various properties by calculating proportions in the following way

$$P[A|B] = \frac{\text{number of individuals satisfying both properties } A \text{ and } B}{\text{number of individuals satisfying property } B}$$

We could approach Example 2.9 in a similar way by creating a "model population" with the appropriate attributes. Since we are given percentages of households with various properties, we can imagine a model population of 100 households, in which 75 have radios (R, 75%), 65 have irons (I), 55 have electric toasters (T), 50 have $I \cap R$, 40 have $R \cap T$, 30 have $I \cap T$, and

20 have all three. The diagram in the solution could be modified by changing the decimals to numbers out of 100, so .2 becomes 20, etc. Then to solve (i), since 55 have toasters and 40 have both a radio and a toaster, the proportion of those who have toasters that also have a radio is $\frac{40}{55} = \frac{8}{11}$.

Creating a model population is sometimes an efficient way of solving a problem involving conditional probabilities, particularly when applying Bayes rule. The following example illustrates this.

Example 2.10. (SOA):
A blood test indicates the presence of a particular disease 95% of the time when the disease is actually present. The same test indicates the presence of the disease 0.5% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

Solution:
We identify the following events:

D : a person has the disease;

TP : a person tests positive for the disease.

We are given

$P[D] = 0.01$, from which we find $P[D'] = 0.99$,

$P[TP|D] = 0.95$, from which we find $P[TP'|D] = 0.05$, and

$P[TP|D'] = 0.005$, from which we find $P[TP'|D'] = 0.995$.

We wish to find $P[D|TP]$. We first solve the problem using rules of conditional probability.

We have $P[D|TP] = \frac{P[D \cap TP]}{P[TP]}$.

We also have, $P[D \cap TP] = P[TP|D] \times P[D] = 0.95 \times 0.01 = 0.0095$, and

$$\begin{aligned} P[TP] &= P[D \cap TP] + P[D' \cap TP] \\ &= P[TP|D] \times P[D] + P[TP|D'] \times P[D'] \\ &= 0.95 \times 0.01 + 0.005 \times .99 = .01445. \end{aligned}$$

Then, $P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{0.0095}{0.01445} = 0.657$.

We can also solve this problem with the model population approach. We imagine a model population of 100,000 individuals. In this population, the number with disease is $\#(D) = 1000$ (.01 of the population) , the number without disease is $\#(D') = 99{,}000$ (0.99 of the population).

Since $P[P|D] = 0.95$, it follows that 95% of those with the disease will test positive, so the number who have the disease and test positive is $\#(TP \cap D) = 0.95 \times 1000 = 950$ (this just reflects the fact that $P[TP \cap D] = P[TP|D] \times P[D] = 0.95 \times 0.01 = .0095$, so that $0.0095 \times 100{,}000 = 950$ in the population have the disease and test positive. In the same way, we find $\#(TP \cap D') = 0.005 \times 99{,}000 = 495$ is the number who do not have disease but test positive. Therefore, the total number who test positive is

$$\#(TP) = \#(TP \cap D) + \#(TP \cap D') = 950 + 495 = 1445.$$

The probability that a person has the disease given that the test indicates the presence of the disease is the proportion that have the disease and test positive as a fraction of all those who test positive,

$$P[D|TP] = \frac{\#(TP \cap D)}{\#(TP)} = \frac{950}{1445} = 0.657.$$

The following table summarizes the calculations in the conditional probability approach.

	$P[D] = 0.01$, given	$P[D'] = 0.99$ $= 1 - 0.01$
TP	$P[TP|D] = 0.95$, given $P[TP \cap D]$ $= P[TP|D] \times P[D]$ $= 0.95 \times 0.01 = 0.0095$	$P[TP|D'] = 0.005$, given $P[TP \cap D']$ $= P[TP|D'] \times P[D']$ $= 0.005 \times 0.99 = 0.00495$
TP'	$P[TP'|D] = 1 - P[TP|D]$ $= 0.05$, $P[TP' \cap D]$ $= P[TP'|D] \times P[D]$ $= 0.05 \times 0.01 = 0.0005$	$P[TP'|D'] = 1 - P[TP|D]$ $= 0.995$, $P[TP' \cap D']$ $= P[TP'|D'] \times P[D']$ $= 0.995 \times 0.99 = 0.98505$

$P[TP] = P[TP \cap D] + P[TP \cap D'] = 0.0095 + 0.00495 = 0.01445.$

$P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{0.0095}{0.01445} = 0.657.$ □

2.4 Problem Set 2

1. Let A, B, C and D be events such that $B = A'$, $C \cap D = \emptyset$, and $P[A] = \frac{1}{4}$, $P[B] = \frac{3}{4}$, $P[C|A] = \frac{1}{2}$, $P[C|B] = \frac{3}{4}$, $P[D|A] = \frac{1}{4}$, $P[D|B] = \frac{1}{8}$. Calculate $P[C \cup D]$.

   (A) $\frac{5}{32}$ (B) $\frac{1}{4}$ (C) $\frac{27}{32}$ (D) $\frac{3}{4}$ (E) 1

2. You are given that $P[A] = .5$ and $P[A \cup B] = .7$.
   Actuary 1 assumes that A and B are independent and calculates $P[B]$ based on that assumption. Actuary 2 assumes that A and B mutually exclusive and calculates $P[B]$ based on that assumption. Find the absolute difference between the two calculated values.

   (A) 0 (B) 0.05 (C) 0.10 (D) 0.15 (E) 0.20

3. (SOA) An actuary studying the insurance preferences of automobile owners makes the following conclusions:

   (i) An automobile owner is twice as likely to purchase collision coverage as disability coverage

   (ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.

   (iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

   What is the probability that an automobile owner purchases neither collision nor disability coverage?

   (A) 0.18 (B) 0.33 (C) 0.48 (D) 0.67 (E) 0.82

4. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl 1. Find the probability that bowl 1 still has 5 black and 5 white balls.

   (A) $\frac{2}{3}$ (B) $\frac{3}{5}$ (C) $\frac{6}{11}$ (D) $\frac{1}{2}$ (E) $\frac{6}{13}$

5. (SOA) An insurance company examines its pool of auto insurance customers and gathers the following information:

   (i) All customers insure at least one car.

   (ii) 70% of the customers insure more than one car.

   (iii) 20% of the customers insure a sports car.

   (iv) Of those customers who insure more than one car, 15% insure a sports car.

   Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

   (A) 0.13 (B) 0.21 (C) 0.24 (D) 0.25 (E) 0.30

6. (SOA) An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims. Calculate the probability that a claim submitted to the insurance company includes operating room charges.

(A) 0.10 (B) 0.20 (C) 0.25 (D) 0.40 (E) 0.80

7. Let A, B and C be events such that $P[A|C] = 0.05$ and $P[B|C] = 0.05$. Which of the following statements must be true?

(A) $P[A \cap B|C] = (.05)^2$ (B) $P[A' \cap B'|C] \geq .90$ (C) $P[A \cup B|C] \leq .05$
(D) $P[A \cup B|C'] \geq 1 - (.05)^2$ (E) $P[A \cup B|C'] \geq .10$

8. A system has two components placed in series so that the system fails if either of the two components fails. The second component is twice as likely to fail as the first. If the two components operate independently, and if the probability that the entire system fails is .28, find the probability that the first component fails.

(A) $\dfrac{0.28}{3}$ (B) 0.10 (C) $\dfrac{0.56}{3}$ (D) 0.20 (E) $\sqrt{0.14}$

9. A ball is drawn at random from a box containing 10 balls numbered sequentially from 1 to 10. Let X be the number of the ball selected, let R be the event that X is an even number, let S be the event that $X \geq 6$, and let T be the event that $X \leq 4$. Which of the pairs (R,S), (R,T), and (S,T) are independent?

(A) (R,S) only (B) (R,T) only (C) (S,T) only
(D) (R,S) and (R,T) only (E) (R,S), (R,T) and (S,T)

10. (SOA) A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

(A) 0.20 (B) 0.25 (C) 0.35 (D) 0.42 (E) 0.57

11. If E_1, E_2 and E_3 are events such that $P[E_1|E_2] = P[E_2|E_3] = P[E_3|E_1] = p$,
$P[E_1 \cap E_2] = P[E_1 \cap E_3] = P[E_2 \cap E_3] = r$, and $P[E_1 \cap E_2 \cap E_3] = s$,
find the probability that at least one of the three events occurs.

(A) $1 - \dfrac{r^3}{p^3}$ (B) $\dfrac{3p}{r} - r + s$ (C) $\dfrac{3r}{p} - 3r + s$
(D) $\dfrac{3p}{r} - 6r + s$ (E) $\dfrac{3r}{p} - r + s$

12. (SOA) A public health researcher examines the medical records of a group of 937 men who died in 1999 and discovers that 210 of the men died from causes related to heart disease. Moreover, 312 of the 937 men had at least one parent who suffered from heart disease, and, of these 312 men, 102 died from causes related to heart disease. Determine the probability that a man randomly selected from this group died of causes related to heart disease, given that neither of his parents suffered from heart disease.

(A) 0.115 (B) 0.173 (C) 0.224 (D) 0.327 (E) 0.514

13. In a T-maze, a laboratory rat is given the choice of going to the left and getting food or going to the right and receiving a mild electric shock. Assume that before any conditioning (in trial number 1) rats are equally likely to go the left or to the right. After having received food on a particular trial, the probability of going to the left and right become 0.6 and 0.4, respectively on the following trial. However, after receiving a shock on a particular trial, the probabilities of going to the left and right on the next trial are 0.8 and 0.2, respectively. What is the probability that the animal will turn left on trial number 2?

(A) 0.1 (B) 0.3 (C) 0.5 (D) 0.7 (E) 0.9

14. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change their choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that they get the door with the prize?

(A) 0 (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

15. (SOA) A doctor is studying the relationship between blood pressure and heartbeat abnormalities in their patients. The doctor tests a random sample of the patients and notes their blood pressures (high, low, or normal) and their heartbeats (regular or irregular). The doctor finds that:

(i) 14% have high blood pressure.

(ii) 22% have low blood pressure.

(iii) 15% have an irregular heartbeat.

(iv) Of those with an irregular heartbeat, one-third have high blood pressure.

(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

(A) 2% (B) 5% (C) 8% (D) 9% (E) 20%

16. (SOA) An insurance company issues life insurance policies in three separate categories: standard, preferred, and ultra-preferred. Of the company's policyholders, 50% are standard, 40% are preferred, and 10% are ultra-preferred. Each standard policy-holder has probability 0.010 of dying in the next year, each preferred policyholder has probability 0.005 of dying in the next year, and each ultra-preferred policyholder has probability 0.001 of dying in the next year. A policyholder dies in the next year. What is the probability that the deceased policyholder was ultra-preferred?

(A) 0.0001 (B) 0.0010 (C) 0.0071 (D) 0.0141 (E) 0.2817

17. (SOA) The probability that a randomly chosen male has a circulation problem is 0.25. Males who have a circulation problem are twice as likely to be smokers as those who do not have a circulation problem. What is the conditional probability that a male has a circulation problem, given that he is a smoker?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{2}{5}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

18. (SOA) A study of automobile accidents produced the following data:

Model Year	Proportion of all vehicles	Probability of involvement in an accident
1997	0.16	0.05
1998	0.18	0.02
1999	0.20	0.03
Other	0.46	0.04

An automobile from one of the model years 1997, 1998, and 1999 was involved in an accident. Determine the probability that the model year of this automobile is 1997.

(A) 0.22 (B) 0.30 (C) 0.33 (D) 0.45 (E) 0.50

19. (SOA) An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16-20	0.06	0.08
21-30	0.03	0.15
31-65	0.02	0.49
66-99	0.04	0.28

A randomly selected driver that the company insures has an accident.

Calculate the probability that the driver was age 16-20.

(A) 0.13 (B) 0.16 (C) 0.19 (D) 0.23 (E) 0.40

20. (SOA) Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;

(ii) 30% of the emergency room patients were serious;

(iii) the rest of the emergency room patients were stable;

(iv) 40% of the critical patients died

(v) 10% of the serious patients died; and

(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

(A) 0.06 (B) 0.29 (C) 0.30 (D) 0.39 (E) 0.64

21. Let A, B and C be mutually independent events such that $P[A] = .5, P[B] = .6$ and $P[C] = .1$. Calculate $P[A' \cup B' \cup C]$.

(A) 0.69 (B) 0.71 (C) 0.73 (D) 0.98 (E) 1.00

22. (SOA) An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

(A) 20 (B) 29 (C) 41 (D) 53 (E) 70

23. (SOA) An actuary studied the likelihood that different types of drivers would be involved in at least one collision during any one-year period. The results of the study are presented below.

Type of driver	Percentage of all drivers	Probability of at least one collision
Teen	8%	0.15
Young Adult	16%	0.08
Midlife	45%	0.04
Senior	31%	0.05
Total	100%	

Given that a driver has been involved in at least one collision in the past year, what is the probability that the driver is a young adult driver?

(A) 0.06 (B) 0.16 (C) 0.19 (D) 0.22 (E) 0.25

24. Urn 1 contains 5 red and 5 blue balls. Urn 2 contains 4 red and 6 blue balls, and Urn 3 contains 3 red balls. A ball is chosen at random from Urn 1 and placed in Urn 2. Then a ball is chosen at random from Urn 2 and placed in Urn 3. Finally, a ball is chosen at random from Urn 3.

Find the probability that all three balls chosen are red.

(A) $\frac{5}{11}$ (B) $\frac{5}{12}$ (C) $\frac{5}{21}$ (D) $\frac{5}{22}$ (E) $\frac{5}{33}$

25. A survey of 1000 Canadian sports fans who indicated they were either hockey fans or lacrosse fans or both, had the following result.

(i) 800 indicated that they were hockey fans
(ii) 600 indicated that they were lacrosse fans

Based on the sample, find the probability that a Canadian sports fan is not a hockey fan given that this fan is a lacrosse fan.

(A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) 1

26. (SOA) An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44 Calculate the number of blue balls in the second urn.

(A) 4 (B) 20 (C) 24 (D) 44 (E) 64

27. (SOA) An actuary is studying the prevalence of three health risk factors, denoted by A, B, and C, within a population of women. For each of the three factors, the probability is 0.1 that a woman in the population has only this risk factor (and no others). For any two of the three factors, the probability is 0.12 that she has exactly these two risk factors (but not the other). The probability that a woman has all three risk factors, given that she has A and B, is $\frac{1}{3}$. What is the probability that a woman has none of the three risk factors, given that she does not have risk factor A?

(A) 0.280 (B) 0.311 (C) 0.467 (D) 0.484 (E) 0.700

Problem Set 2 Solutions

1. Since C and D have empty intersection, $P[C \cup D] = P[C] + P[D]$.

   Also, since A and B are "exhaustive" events (since they are complementary events, their union is the entire sample space, with a combined probability of $P[A \cup B] = P[A] + P[B] = 1$).

   We use the rule $P[C] = P[C \cap A] + P[C \cap A']$, and the rule $P[C|A] = \frac{P[A \cap C]}{P[A]}$ to get

   $P[C] = P[C|A] \times P[A] + P[C|A'] \times P[A'] = \frac{1}{2} \times \frac{1}{4} + \frac{3}{4} \times \frac{3}{4} = \frac{11}{16}$ and

   $P[D] = P[D|A] \times P[A] + P[D|A'] \times P[A'] = \frac{1}{4} \times \frac{1}{4} + \frac{1}{8} \times \frac{3}{4} = \frac{5}{32}$.

   Then, $P[C \cup D] = P[C] + P[D] = \frac{27}{32}$.

   Answer C

2. Actuary 1: Since A and B are independent, so are A' and B'.

   $P[A' \cap B'] = 1 - P[A \cup B] = 0.3$.

   But $0.3 = P[A' \cap B'] = P[A'] \times P[B'] = 0.5 \times P[B'] \;\rightarrow\; P[B'] = 0.6 \;\rightarrow\; P[B] = 0.4$.

   Actuary 2: $0.7 = P[A \cup B] = P[A] + P[B] = 0.5 + P[B] \;\rightarrow\; P[B] = 0.2$.

   Absolute difference is $|0.4 - 0.2| = 0.2$.

   Answer E

3. We identify the following events:

   $D =$ an automobile owner purchases disability coverage, and

   $C =$ an automobile owner purchases collision coverage.

   We are given that

   (i) $P[C] = 2P[D]$,

   (ii) C and D are independent, and (iii) $P[C \cap D] = 0.15$.

   From (ii) it follows that $P[C \cap D] = P[C] \times P[D]$, and therefore,

   $0.15 = 2P[D] \times P[D] = 2(P[D])^2$, from which we get $P[D] = \sqrt{0.075} = 0.27386$. Then,

   $P[C] = 2P[D] = 0.54772$, $P[D'] = 1 - P[D] = 0.72614$, and $P[C'] = 1 - P[C] = 0.45228$.

   Since C and D are independent, so are C' and D', and therefore, the probability that an automobile owner purchases neither disability coverage nor collision coverage is $P[C' \cap D'] = P[C'] \times P[D'] = 0.328$.

   Answer B

4. Let C be the event that bowl 1 has 5 black balls after the exchange.

   Let B_1 be the event that the ball chosen from bowl 1 is black, and let B_2 be the event that the ball chosen from bowl 2 is black.

   Event C is the disjoint union of $B_1 \cap B_2$ and $B_1' \cap B_2'$ (black-black or white-white picks), so that $P[C] = P[B_1 \cap B_2] + P[B_1' \cap B_2']$.

The black-black combination has probability $\frac{6}{11} \times \frac{1}{2}$, since there is a $\frac{5}{10}$ chance of picking black from bowl 1, and then (with 6 black in bowl 2, which now has 11 balls) $\frac{6}{11}$ is the probability of picking black from bowl 2. This is

$P[B_1 \cap B_2] = P[B_2|B_1] \times P[B_1] = \frac{6}{11} \times \frac{1}{2}$.

In a similar way, the white-white combination has probability $\frac{6}{11} \times \frac{1}{2}$.

Then $P[C] = \frac{6}{11} \times \frac{1}{2} + \frac{6}{11} \times \frac{1}{2} = \frac{6}{11}$.

Answer C

5. We identify the following events:

A - the policyholder insures exactly one car (so that A' is the event that the policyholder insures more than one car), and

S - the policyholder insures a sports car.

We are given $P[A'] = 0.7$ (from which it follows that $P[A] = 0.3$) , and $P[S] = 0.2$

(and $P[S'] = 0.8$). We are also given the conditional probability $P[S|A'] = 0.15$;

"of those customers who insure more than one car", means we are looking at a conditional event given A'.

We are asked to find $P[A \cap S']$.

We create the following probability table, with the numerals in parentheses indicating the order in which calculations are performed.

	$A, 0.3$	$A', 0.7$
$S, 0.2$	$(\mathbf{1}) P[S \cap A]$ $= P[S] - P[S \cap A']$ $= 0.2 - 0.105 = 0.095$	$(\mathbf{2}) P[S \cap A'] = P[S|A'] \times P[A']$ $= 0.15 \times 0.7 = 0.105$
$S', 0.8$	$(\mathbf{3}) P[A \cap S']$ $= P[A] - P[A \cap S]$ $= 0.3 - 0.095 = 0.205$	

We can solve this problem with a model population of 1000 individuals with auto insurance.

$\#A = 300$ (since 70% insure more than one car), and $\#S = 200$.
From $P[S|A'] = 0.15$ we get

$\#S \cap A' = 0.15 \times \#A' = 0.15 \times 700 = 105$. Then $\#S \cap A = \#S - \#S \cap A' = 200 - 105 = 95$,

and $\#S' \cap A = \#A - \#S \cap A = 300 - 95 = 205$ is the number that insure exactly one car and the car is not a sports car.

Therefore $P[S' \cap A] = 0.205$.

Answer B

6. We define the following events.

   E - the claim includes emergency room charges,

   O - the claim includes operating room charges.

   We are given $P[E \cup O] = 0.85$, $P[E'] = 0.25$ and E and O are independent.

   We are asked to find $P[O]$.

   We use the probability rule $P[E \cup O] = P[E] + P[O] - P[E \cap O]$.

   Since E and O are independent, we have $P[E \cap O] = P[E] \times P[O] = 0.75 \times P[O]$ (since $P[E] = 1 - P[E'] = 1 - 0.25 = 0.75$).

   Therefore, $0.85 = P[E \cup O] = 0.75 + P[O] - 0.75 \times P[O]$.

   Solving for $P[O]$ results in $P[O] = 0.40$.

   Answer D

7. $P[A' \cap B'|C] = P[(A \cup B)'|C] = 1 - P[A \cup B|C] \geq 0.9$,

   since $P[A \cup B|C] \leq P[A|C] + P[B|C] = 0.1$.

   Answer B

8. $0.28 = P[C_1 \cup C_2] = P[C_1] + P[C_2] - P[C_1 \cap C_2] = P[C_1] + 2P[C_1] - 2(P[C_1])^2$

   Solving the quadratic equation results in $P[C_1] = 0.1$ (or 1.4, but we disregard this solution since $P[C_1]$ must be ≤ 1). Alternatively, each of the five answers can be substituted into the expression above for $P[C_1]$ to see which one satisfies the equation.

   Answer B

9. $P[R] = 0.5$, $P[S] = 0.5$, $P[T] = 0.4$.

   $P[R \cap S] = P[6,8,10] = 0.3 \neq 0.5 \times .5 = P[R] \times P[S] \ \rightarrow R,S$ are not independent

   $P[R \cap T] = P[2,4] = 0.2 = 0.5 \times 0.4 = P[R] \times P[T] \ \rightarrow \ R,T$ are independent

   $P[S \cap T] = P[\emptyset] = 0 \neq 0.5 \times 0.4 = P[S] \times P[T] \rightarrow S,T$ are not independent.

   Answer B

10. We identify the following events

    N - non-smoker,

    L - light smoker,

    H - heavy smoker,

    D - dies during the 5-year study.

    We are given $P[N] = 0.50\,, P[L] = 0.30\,, P[H] = 0.20\,.$

    We are also told that $P[D|L] = 2 \times P[D|N] = \frac{1}{2} \times P[D|H]$

    (the probability that a light smoker dies during the 5-year study period is $P[D|L]$;

it is the conditional probability of dying during the period given that the individual is a light smoker). We wish to find the conditional probability $P[H|D]$.We will find this probability from the basic definition of conditional probability, $P[H|D]=\frac{P[H\cap D]}{P[D]}$. These probabilities can be found from the following probability table. The numerals indicate the order in which the calculations are made. We are not given specific values for $P[D|L]$, $P[D|N]$, or $P[D|H]$, so will let $P[D|N]=k$,
and then $P[D|L]=2k$ and $P[D|H]=4k$.

	$N, 0.5$	$L, 0.3$	$H, 0.2$
D	**(1)**$P[D\cap N]$	**(2)**$P[D\cap L]$	**(3)**$P[D\cap H]$
	$=P[D|N]\times P[N]$	$=P[D|L]\times P[L]$	$=P[D|H]\times P[H]$
	$=k\times 0.5=0.5k$	$=2k\times 0.3=0.6k$	$4k\times 0.2=0.8k$

(4) $P[D]=P[D\cap N]+P[D\cap L]+P[D\cap H]=0.5k+0.6k+0.8k=1.9k$

(5) $P[H|D]=\frac{P[H\cap D]}{P[D]}=\frac{0.8k}{1.9k}=0.42.$

Answer D

11. $P[E_1|E_2]=\frac{P[E_1\cap E_2]}{P[E_2]}=p \ \rightarrow \ P[E_2]=\frac{r}{p}$, and similarly $P[E_3]=P[E_1]=\frac{r}{p}$.

Then, $P[E_1\cup E_2\cup E_3]$

$=P[E_1]+P[E_2]+P[E_3]-(P[E_1\cap E_2]+P[E_1\cap E_3]+P[E_2\cap E_3])+P[E_1\cap E_2\cap E_3]$

$=3\times\frac{r}{p}-3r+s.$

Answer C

12. In this group of 937 men, we regard proportions of people with certain conditions to be probabilities. We are given the population of 937 men. We identify the following conditions:

DH - died from causes related to heart disease, and

PH - had a parent with heart disease.

We are given $\#PH=312$, so if follows that $\#PH'=937-312=625$.

We are also given $\#DH=210$ and $\#DH\cap PH=102$.

It follows that $\#DH\cap(PH')=\#DH-\ \#DH\cap PH=210-102=108$.

Then the probability of dying due to heart disease given that neither parent suffered from heart disease is the proportion $\frac{\#DH\cap(PH')}{\#PH'}=\frac{108}{625}$. The solution in terms of conditional probability rules is as follows. From the given information, we have

$P[DH]=\frac{210}{937}$ (proportion who died from causes related to heart disease)

$P[PH]=\frac{312}{937}$ (proportion who have parent with heart disease)

$P[DH|PH]=\frac{102}{312}$ (prop. who died from heart disease given that a parent has heart disease).

We are asked to find $P[DH|PH']$ (PH' is the complement of event PH, so that PH' is the event that neither parent had heart disease). Using event algebra, we have

$P[DH|PH] = \frac{P[DH \cap PH]}{P[PH]} \Rightarrow P[DH \cap PH] = P[DH|PH] \times P[PH] = \frac{102}{312} \times \frac{312}{937} = \frac{102}{937}$.

We now use the rule $P[A] = P[A \cap B] + P[A \cap \overline{B}]$.

Then $P[DH] = P[DH \cap PH] + P[DH \cap PH'] \rightarrow \frac{210}{937} = \frac{102}{937} + P[DH \cap PH']$

$\Rightarrow P[DH \cap PH'] = \frac{108}{937}$.

Finally, $P[DH|PH'] = \frac{P[DH \cap PH']}{P[PH']} = \frac{108/937}{1-P[PH]} = \frac{108/937}{1-\frac{312}{937}} = \frac{108}{625} = 0.1728$.

These calculations can be summarized in the following table.

	$DH, 210$ (given)	$DH', 727$ (given)
$PH, 312$ (given)	$DH \cap PH = 102$ (given)	$DH' \cap PH = 210$ $= 312 - 102$
$PH', 625$ $= 937 - 312$	$DH \cap PH' = 108$ $210 - 102$	$DH' \cap PH' = 517$ $727 - 210$ $625 - 108$

Hence, $P[DH|PH'] = \frac{P[DH \cap PH']}{P[PH']} = \frac{\#[DH \cap PH']}{\#[PH']} = \frac{108}{625} = 0.1728$.

In this example, probability of an event is regarded as the proportion of a group that experiences that event.

Answer B

13. $L1$ = turn left on trial 1, $R1$ = turn right on trial 1, $L2$ = turn left on trial 2.

We are given that $P[L1] = P[R1] = 0.5$.

$P[L2] = P[L2 \cap L1] + P[L2 \cap R1]$ since $L1$,$R1$ form a partition.

$P[L2|L1] = 0.6$ (if the rat turns left on trial 1 then it gets food and has a 0.6 chance of turning left on trial 2). Then $P[L2 \cap L1] = P[L2|L1] \times P[L1] = 0.6 \times 0.5 = 0.3$.

In a similar way, $P[L2 \cap R1] = P[L2|R1] \times P[R1] = 0.8 \times 0.5 = 0.4$.

Then, $P[L2] = 0.3 + 0.4 = 0.7$.

In a model population of 10 rats, $\#L1 = \#R1 = 5$, and $\#L2 \cap L1 = 0.6 \times 5 = 3$

and $\#L2 \cap R1 = 0.8 \times 5 = 4$. Then the number turning left on trial 2 will be

$\#L2 = \#L2 \cap L1 + \#L2 \cap R1 = 3 + 4 = 7$,

so the probability of a rat turning left on trial 2 is $7/10 = 0.7$.

Answer D

14. We define the events $A =$ prize door is chosen after contestant switches doors,

$B =$ prize door is initial one chosen by contestant. Then $P[B] = \frac{1}{3}$, since each door is equally likely to hold the prize initially. To find $P[A]$ we use the Law of Total Probability.

$$P[A] = P[A|B] \times P[B] + P[A|B'] \times P[B'] = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}.$$

If the prize door is initially chosen, then after switching, the door chosen is not the prize door, so that $P[A|B] = 0$. If the prize door is not initially chosen, then since the host shows the other non-prize door, after switching the contestant definitely has the prize door, so that $P[A|B'] = 1$.

Answer E

15. This question can be put into the context of probability event algebra. First we identify events:

$H =$ high blood pressure,

$L =$ low blood pressure,

$N =$ normal blood pressure,

$R =$ regular heartbeat,

$I = R' =$ irregular heartbeat

We are told that 14% of patients have high blood pressure, which can be represented as $P[H] = 0.14$, and similarly $P[L] = .22$, and therefore $P[N] = 1 - P[H] - P[L] = 0.64$.

We are given $P[I] = 0.15$, so that $P[R] = 1 - P[I] = 0.85$.

We are told that "of those with an irregular heartbeat, one-third have high blood pressure". This is the conditional probability that given I (irregular heartbeat) the probability of H (high blood pressure) is $P[H|I] = \frac{1}{3}$. Similarly, we are given $P[I|N] = \frac{1}{8}$.

We are asked to find the portion of patients who have both a regular heartbeat and low blood pressure; this is $P[R \cap L]$. Since every patient is exactly one of H, L or N, we have

$P[R \cap L] + P[R \cap H] + P[R \cap N] = P[R] = 0.85$, so that

$P[R \cap L] = 0.85 - P[R \cap H] - P[R \cap N]$.

From the conditional probabilities we have

$\frac{1}{3} = P[H|I] = \frac{P[H \cap I]}{P[I]} = \frac{P[H \cap I]}{0.15} \rightarrow P[H \cap I] = 0.05$, and

$\frac{1}{8} = P[I|N] = \frac{P[I \cap N]}{P[N]} = \frac{P[I \cap N]}{0.64} \rightarrow P[I \cap N] = 0.08$.

Then, since all patients are exactly one of I and R, we have

$P[H \cap I] + P[H \cap R] = P[H] = 0.14 \rightarrow P[H \cap R] = 0.14 - 0.05 = 0.09$, and

$P[I \cap N] + P[R \cap N] = P[N] = 0.64 \rightarrow P[R \cap N] = 0.64 - 0.08 = 0.56$.

Finally, $P[R \cap L] = .85 - P[R \cap H] - P[R \cap N] = 0.85 - 0.09 - 0.56 = 0.20$.

Note that the entries $P[R \cap H]$ and $P[R \cap N]$ can also be calculated from this table.

The model population solution is as follows. Suppose that the model population has 2400 individuals. Then we have the following

$\#H = 0.14 \times 2400 = 336$, $\#L = 528$, $\#N = 1536$, $\#I = 360$, $\#R = 2040$.

Since one-third of those with an irregular heartbeat have high blood pressure, we get

$\#I \cap H = 120$, and since one-eighth of those with normal blood pressure have an irregular heartbeat we get $\#N \cap I = 192$. We wish to find $\#R \cap L$.

From $\#I = \#I \cap H + \#I \cap L + \#I \cap N$, we get $360 = 120 + \#I \cap L + 192$,

so that $\#I \cap L = 48$. Then from $\#L = \#I \cap L + \#R \cap L$ we get $528 = 48 + \#R \cap L$,

so that $\#R \cap L = 480$.

Finally, the probability of having a regular heartbeat and low blood pressure is the proportion of the population with those properties, which is $\frac{480}{2400} = 0.2$.

Answer E

16. This is a typical exercise involving conditional probability.

We first label the events, and then identify the probabilities.

S - standard policy

P - preferred policy

U - ultra-preferred policy

D - death occurs in the next year.

We are given

$P[S] = 0.50\,, P[P] = 0.40\,, P[U] = 0.10\,,$
$P[D|S] = 0.01\,, P[D|P] = 0.005\,, P[D|U] = 0.001.$

We are asked to find $P[U|D]$.

The model population solution is as follows. Suppose there is a model population of 10,000 insured lives.

Then $\#S = 5000$, $\#P = 4000$ and $\#U = 1000$.

From $P[D|S] = 0.01$ we get $\#D \cap S = .01 \times 5000 = 50$, and we also get

$\#D \cap P = 0.005 \times 4000 = 20$ and $\#D \cap U = 0.001 \times 1000 = 1$.

Then $\#D = 50 + 20 + 1 = 71$, and $P[U|D]$ is the proportion who are ultra-preferred as a proportion of all who died. This is $\frac{1}{71} = 0.0141$.

The conditional probability approach to solving the problem is as follows.

The basic formulation for conditional probability is $P[U|D] = \frac{P[U \cap D]}{P[D]}$.

We use the following relationships:

$P[A \cap B] = P[A|B] \times P[B]$, and

$P[A] = P[A \cap C_1] + P[A \cap C_2] + \cdots + P[A \cap C_n]$, for a partition $C_1, C_2 \ldots, C_n$.

In this problem, events S, P and U form a partition of all policyholders.

Using the relationships we get

$P[U \cap D] = P[D|U] \times P[U] = 0.001 \times 0.1 = 0.0001$, and

$$\begin{aligned} P[D] &= P[D \cap S] + P[D \cap P] + P[D \cap U] \\ &= P[D|S] \times P[S] + P[D|P] \times P[P] + P[D|U] \times P[U] \\ &= 0.01 \times 0.5 + 0.005 \times 0.4 + 0.001 \times 0.1 = 0.0071. \end{aligned}$$

Then,

$$P[U|D] = \frac{P[U \cap D]}{P[D]} = \frac{P[D|U] \times P[U]}{P[D|S] \times P[S] + P[D|P] \times P[P] + P[D|U] \times P[U]}$$

$$= \frac{0.001 \times 0.1}{0.01 \times 0.5 + 0.005 \times 0.4 + 0.001 \times 0.1} = \frac{0.0001}{0.0071} = 0.0141.$$

Notice that the numerator is one of the factors of the denominator. This will always be the case when we are "reversing" conditional probabilities such as has been done here; we are to find $P[U|D]$ from being given information about $P[D|U]$, $P[D|S]$, $P[D|P]$, etc.

From the calculations already made it is easy to find the probability that the deceased policyholder was preferred;

$$P[P|D] = \frac{P[P \cap D]}{P[D]} = \frac{P[D|P] \times P[P]}{P[D|S] \times P[S] + P[D|P] \times P[P] + P[D|U] \times P[U]}$$

$$= \frac{0.005 \times 0.4}{0.01 \times 0.5 + 0.005 \times 0.4 + 0.001 \times 0.1} = \frac{0.0020}{0.0071} = 0.2817.$$

And $P[S|D]$ is

$$\frac{0.01 \times 0.5}{.01 \times 0.5 + 0.005 \times 0.4 + 0.00 \times 0.1} = \frac{0.0050}{0.0071} = 0.7042.$$

The calculations can be summarized in the following table:

	$S, 0.5$ given	$P, 0.4$ given	$U, 0.1$ given
D	$P(D\|S) = 0.01$ given	$P(D\|P) = 0.005$ given	$P(D\|U) = 0.001$ given
	$P(D \cap S)$ $= P(D\|S) \times P(S)$ $= 0.01 \times 0.5 = 0.005$	$P(D \cap P)$ $= P(D\|P) \times P(P)$ $= 0.005 \times 0.4 = 0.002$	$P(D \cap U)$ $P(D\|U) \times P(U)$ $= 0.001 \times 0.1 = 0.0001$

$P[D] = P[D \cap S] + P[D \cap P] + P[D \cap U] = 0.005 + 0.002 + 0.0001 = 0.0071.$

$P[U|D] = \frac{P[U \cap D]}{P[D]} = \frac{0.0001}{0.0071} = 0.0141.$

Answer D

17. We identify the following events:

C - a randomly chosen male has a circulation problem ,

S - a randomly chosen male is a smoker.

We are given the following probabilities:

$P[C] = 0.25$, $P[S|C] = 2P[S|C']$.

From the rule $P[A \cap B] = P[A|B]P[B]$, we get

$P[S \cap C] = P[S|C] \times P[C] = 0.25 \times P[S|C]$, and

$P[S \cap C'] = P[S|C'] \times P[C'] = P[S|C'] \times (1 - P[C]) = 0.75 \times \frac{1}{2} \times P[S|C]$,

so that $P[S] = P[S \cap C] + P[S \cap C'] = 0.25 \times P[S|C] + 0.75 \times \frac{1}{2} \times P[S|C] = 0.625 \times P[S|C]$.

We are asked to find $P[C|S]$. This is $P[C|S] = \frac{P[C \cap S]}{P[S]} = \frac{0.25 \times P[S|C]}{0.625 \times P[S|C]} = 0.4$.

Note that the way in which information was provided allowed us to formulate various probabilities in terms of $P[S|C]$ (but we do not have enough to find $P[S|C]$).

Answer C

18. We identify events as follows:

97: the model year is 1997; 98: the model year is 1998; 99: the model year is 1999;

OO : other the model year is not 1997, 1998 or 1999; A : the car is involved in an accident

We are given $P[97] = 0.16$, $P[98] = 0.18$, $P[99] = 0.20$, $P[OO] = 0.46$,

$P[A|97] = 0.05$, $P[A|98] = 0.02$, $P[A|99] = 0.03$, $P[A|\text{other}] = 0.04$.

The model population solution is as follows. Suppose there are 10,000 automobiles in the study. Then $\#97 = 1600$, $\#98 = 1800$, $\#99 = 2000$, $\#OO = 4600$.

From $P[A|97] = .05$ we get $\#A \cap 97 = .05 \times 1600 = 80$, and in a similar way we get

$\#A \cap 98.02 \times 1800 = 36$, $\#A \cap 99 = .03 \times 2000 = 60$

and $\#A \cap OO = .04 \times 4600 = 184$.

We are given that an automobile from one of 97, 98 or 99 was involved in an accident, and we wish to find the probability that it was a 97 model. This is the conditional probability $P[97|A \cap (97 \cup 98 \cup 99)]$. This will be the proportion

$$\frac{\#A \cap 97}{\#A \cap 97 + \#A \cap 98 + \#A \cap 99} = \frac{80}{80 + 36 + 60} = \frac{80}{176} = 0.4545.$$

The conditional probability approach to solve the problem is as follows.

We use the conditional probability rule $P[C|D] = \frac{P[C \cap D]}{P[D]}$, so that

$$P[97|A \cap (97 \cup 98 \cup 99)] = \frac{P[97 \cap [A \cap (97 \cup 98 \cup 99)]]}{P[A \cap (97 \cup 98 \cup 99)]}.$$

From set algebra, we have $97 \cap [A \cap (97 \cup 98 \cup 99)] = 97 \cap A$, and

$A \cap (97 \cup 98 \cup 99) = (A \cap 97) \cup (A \cap 98) \cup (A \cap 99)$.

Since the events 97, 98 and 99 are disjoint, we get

$P[A \cap (97 \cup 98 \cup 99)] = P[(A \cap 97) \cup (A \cap 98) \cup (A \cap 99)] = P[A \cap 97] + P[A \cap 98] + P[A \cap 99]$.

From conditional probability rules we have

$P[A \cap 97] = P[A|97] \times P[97] = 0.05 \times 0.16 = 0.008$, and similarly

$P[A \cap 98] = 0.02 \times 0.18 = 0.0036$, and $P[A \cap 99] = 0.03 \times 0.20 = 0.006$.

Then, $P[A \cap (97 \cup 98 \cup 99)] = 0.008 + 0.0036 + 0.006 = 0.0176$.

Therefore, the probability we are trying to find is

$$P[97|A \cap (97 \cup 98 \cup 99)] = \frac{P[97 \cap [A \cap (97 \cup 98 \cup 99)]]}{P[A \cap (97 \cup 98 \cup 99)]}$$

$$= \frac{P[97 \cap A]}{P[A \cap (97 \cup 98 \cup 99)]} = \frac{0.008}{0.0176} = 0.4545.$$

These calculations can be summarized in the following table:

	$97, 0.16$ given	$98, 0.18$ given	$99, 0.20$ given	Other, 0.46 given
A	$P(A\|97)$ $= 0.05$ given	$P(A\|98)$ $= 0.02$ given	$P(A\|99)$ $= 0.03$ given	$P(A\|\text{Other})$ $= 0.04$ given
	$P(A \cap 97)$ $P[A\|97] \times P[97]$ $= 0.05 \times 0.16$ $= 0.008$	$P(A \cap 98)$ $P[A\|98] \times P[98]$ $= 0.02 \times 0.18$ $= 0.0036$	$P(A \cap 99)$ $P[A\|99] \times P[99]$ $= 0.03 \times 0.20$ $= 0.006$	$P(A \cap 98)$ $P[A\|O] \times P[O]$ $= 0.04 \times 0.46$ $= 0.0184$

Then, $P[97|A \cap (97 \cup 98 \cup 99)] = \frac{0.008}{0.008+0.0036+0.006} = 0.4545$.

Note that the denominator is the sum of the first three of the intersection probabilities, since the condition is that the auto was 97, 98 or 99. If the question had asked for the probability that the model year was 97 given that an accident occurred (without restricting to 97, 98, 99) then the probability would be $\frac{0.008}{0.008+0.0036+0.006+0.0184}$; we would include all model years in the denominator. If the question had asked for the probability that the model year was 97 given that an accident occurred and the automobile was from one of the model years 97 or 98, then the probability would be $\frac{0.008}{0.008+0.0036}$; we would include only the 97 and 98 model years.

Answer D

19. We identify the following events:

A - the driver has an accident,

T (teen) - age of driver is 16-20,

Y (young) - age of driver is 21-30,

M (middle age) - age of driver is 31-65,

S (senior) - age of driver is 66-99.

The final column in the table lists the probabilities of T, Y, M and S, and the middle column gives the conditional probability of A given driver age. The table can be interpreted as

Age	Probability of Accident	Portion of Insured Drivers
16-20	$P[A\|T] = 0.06$	$P[T] = 0.08$
21-30	$P[A\|Y] = 0.03$	$P[Y] = 0.15$
31-65	$P[A\|M] = 0.02$	$P[M] = 0.49$
66-99	$P[A\|S] = 0.04$	$P[S] = 0.28$

We are asked to find $P[T|A]$.

We construct the following probability table, with numerals in parentheses indicating the order of the calculations.

	$T, 0.08$	$Y, 0.15$	$M, 0.49$	$S, 0.28$
A	**(1)**$P[A\cap T]$ $= P[A\|T]\times P[T]$ $= 0.06\times 0.08$ $= 0.0048$	**(2)**$P[A\cap Y]$ $= P[A\|Y]\times P[Y]$ $= 0.03\times 0.15$ $= 0.0045$	**(3)**$P[A\cap M]$ $= P[A\|M]\times P[M]$ $= 0.02\times 0.49$ $= 0.0098$	**(4)**$P[A\cap S]$ $= P[A\|S]\times P[S]$ $= 0.04\times 0.28$ $= 0.0112$

(5) $P[A] = P[A\cap T + P[A\cap Y] + P[A\cap M] + P[A\cap S] = 0.0303$

(6) $P[T|A] = \frac{P[T\cap A]}{P[A]} = \frac{0.0048}{0.0303} = 0.158.$

Answer B

20. We label the following events:

C - critical , S - serious , T - stable , D - died , D' - survived.

The following information is given

$P[C] = 0.1,\ P[S] = 0.3,\ P[T] = 0.6 = 1 - P[C] - P[S],\ \ P[D|C] = 0.4,$
$P[D|S] = 0.1, P[D|T] = 0.01.$

We are asked to find $P[S|D']$. This can be done by using the table of probabilities found below. The rules being used here are $P[A\cap B] = P[A|B]\times P[B]$,
and $P[A] = P[A\cap B_1] + P[A\cap B_2] + \cdots + P[A\cap B_n]$ if $B_1, B_2, \ldots, B_n$ form a partition of the probability space. In this case, C, S, T form a partition since all patients are exactly one of these three conditions.

	C	S	T
D	$P[D\cap C]$ $=P[D\vert C]\times P[C]$ $=0.4\times 0.1=0.04$	$P[D\cap S]$ $=P[D\vert S]\times P[S]$ $=0.1\times 0.3=0.03$	$P[D\cap T]$ $=P[D\vert T]\times P[T]$ $=0.1\times 0.6=0.006$
	$\to P[D]=P[D\cap C]+P[D\cap S]+P[D\cap T]=0.04+0.03+0.006=0.076$		
D'	$P[D'\cap C]$ $=P[D'\vert C]\times P[C]$ $=0.6\times 0.1=0.06$	$P[D'\cap S]$ $=P[D'\vert S]\times P[S]$ $=0.9\times 0.3=0.27$	$P[D'\cap T]$ $=P[D'\vert T]\times P[T]$ $=0.99\times 0.6=0.594$
	$\to P[D']=P[D'\cap C]+P[D'\cap S]+P[D'\cap T]=0.06+0.27+0.594=0.924$		

It was not necessary to do the calculations for D', since
$P[D']=1-P[D]=1-0.076=0.924$.

The probability in question is $P[S|D']=\frac{P[S\cap D']}{P(D')}=\frac{0.27}{0.924}=0.292$.

Answer B

21. $P[A'\cup B'\cup C]$

$=P[A']+P[B']+P[C]-(P[A'\cap B']+P[A'\cap C]+P[B'\cap C])+P[A'\cap B'\cap C]$

$=0.5+0.4+0.1-[0.5\times 0.4+0.5\times 0.1+0.4\times 0.1]+0.5\times 0.4\times 0.1=0.73.$

If events X and Y are independent, then so are X' and Y, X and Y', and X' and Y'. Alternatively using DeMorgan's Law, we have

$$P[A'\cup B'\cup C]=1-P[(A'\cup B'\cup C)']=1-P[A''\cap B''\cap C']=1-P[A\cap B\cap C']$$
$$=1-P[A]\times P[B]\times P[C']=1-0.5\times 0.6\times 0.9=0.73.$$

Answer C

22. We define the following events

R - renew at least one policy next year

A - has an auto policy , H - has a homeowner policy

A policyholder with an auto policy only can be described by the event $A\cap H'$, and a policyholder with a homeowner policy only can be described by the event $A'\cap H$.

We are given $P[R|A\cap H']=0.4$, $P[R|A'\cap H]=0.6$ and $P[R|A\cap H]=0.8$.

We are also given $P[A]=0.65$, $P[H]=0.5$ and $P[A\cap H]=0.15$.

We are asked to find $P[R]$.

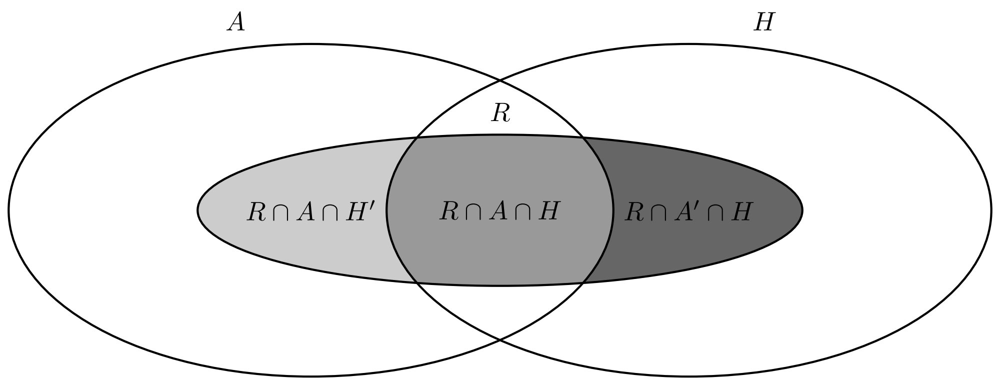
We use the rule $P[R]=P[R\cap A\cap H]+P[R\cap A'\cap H]+P[R\cap A\cap H']+P[R\cap A'\cap H']$.

Since renewal can only occur if there is at least one policy, it follows that $P[R \cap A' \cap H'] = 0$; in other words, if there is no auto policy (event A') and there is no homeowner policy (event H'), then there can be no renewal. An alternative way of saying the same thing is that R (the inside oval in the diagram) is a subset (subevent) of $A \cup H$.

(Note also that $P[A \cup H] = P[A] + P[H] - P[A \cap H] = 0.65 + 0.5 - 0.15 = 1$, so this also show that R must be a subevent of $A \cup H$, and it also shows that

$P[A' \cap H'] = 1 - P[A \cup H] = 1 - 1 = 0$ so that $A' \cap H' = \phi$).

This can be illustrated in the following diagram.



We find $P[R \cap A \cap H]$, $P[R \cap A' \cap H]$ and $P[R \cap A \cap H']$ by using the rule

$P[C \cap D] = P[C|D] \times P[D]$:

$P[R \cap A \cap H] = P[R|A \cap H] \times P[A \cap H] = 0.8 \times 0.15 = 0.12$,

$P[R \cap A' \cap H] = P[R|A' \cap H] \times P[A' \cap H] = 0.6 \times P[A' \cap H]$,

$P[R \cap A \cap H'] = P[R|A \cap H'] \times P[A \cap H'] = 0.4 \times P[A \cap H']$.

In order to complete the calculations we must find $P[A' \cap H]$ and $P[A \cap H']$.

From the diagram above, or using the probability rule, we have

$P[A] = P[A \cap H] + P[A \cap H'] \ \rightarrow \ 0.65 = 0.15 + P[A \cap H'] \ \rightarrow \ P[A \cap H'] = 0.5$, and

$P[H] = P[A \cap H] + P[A' \cap H] \ \rightarrow \ 0.5 = 0.15 + P[A' \cap H] \ \rightarrow \ P[A' \cap H] = 0.35$.

Then $\quad P[R \cap A' \cap H] = 0.6 \times 0.35 = 0.21$ and $\ P[R \cap A \cap H'] = 0.4 \times 0.5 = 0.2$.

Finally, $\ P[R] = 0.12 + 0.21 + 0.2 = 0.53$. 53% of policyholders will renew.

Answer D

23. We are given $P[\text{teen}] = 0.08$, $P[\text{young adult}] = 0.16$, $P[\text{midlife}] = 0.45$ and

$P[\text{senior}] = 0.31$. We are also given the conditional probabilities

$P[\text{at least one collision}|\text{teen}] = 0.15$, $P[\text{at least one collision}|\text{young adult}] = 0.08$,

$P[\text{at least one collision}|\text{midlife}] = 0.04$, $P[\text{at least one collision}|\text{senior}] = 0.05$.

We wish to find $P[\text{young adult}|\text{at least one collision}]$.

Using the definition of conditional probability, we have

$$P[\text{young adult}|\text{at least one collision}] = \frac{P[\text{young adult} \cap \text{at least one collision}]}{P[\text{at least one collision}]}.$$

We use the rule $P[A \cap B] = P[A|B] \times P[B]$, to get

$$P[\text{young adult} \cap \text{at least one collision}] = P[\text{at least one collision} \cap \text{young adult}]$$
$$= P[\text{at least one collision}|\text{young adult}] \times P[\text{young adult}] = 0.08 \times 0.16 = 0.0128.$$

We also have

$$\begin{aligned} P[\text{at least one collision}] &= P[\text{at least one collision} \cap \text{teen}] \\ &\quad + P[\text{at least one collision} \cap \text{young adult}] \\ &\quad + P[\text{at least one collision} \cap \text{midlife}] \\ &\quad + P[\text{at least one collision} \cap \text{senior}] \\ &= P[\text{at least one collision}|\text{teen}] \times P[\text{teen}] \\ &\quad + P[\text{at least one collision}|\text{young adult}] \times P[\text{young adult}] \\ &\quad + P[\text{at least one collision}|\text{midlife}] \times P[\text{midlife}] \\ &\quad + P[\text{at least one collision}|\text{senior}] \times P[\text{senior}] \\ &= 0.15 \times 0.08 + 0.08 \times 0.16 + 0.04 \times 0.45 + 0.05 \times 0.31 = 0.0583 \end{aligned}$$

Then $P[\text{young adult}|\text{at least one collision}] = \frac{0.0128}{0.0583} = 0.2196$.

	$T, 0.08$ given	$Y, 0.16$ given	$M, 0.45$ given	$S, 0.31$ given
At least one collision	$P[C\|T]$ $= 0.15$ given	$P[C\|Y]$ $= 0.08$ given	$P[C\|M]$ $= 0.04$ given	$P[C\|S]$ $= 0.05$ given
	$P[C \cap T]$ $= 0.15 \times 0.28$ $= 0.012$	$P[C \cap Y]$ $= 0.18 \times 0.16$ $= 0.0128$	$P[C \cap M]$ $= 0.04 \times 0.45$ $= 0.018$	$P[C \cap S]$ $= 0.05 \times 0.31$ $= 0.0155$

$$P[\text{at least one collision}] = P[C] = P[C \cap T] + P[C \cap Y] + P[C \cap M] + P[C \cap S]$$
$$= .012 + .0128 + .018 + .0155 = .0583.$$

$$P[\text{young adult}|\text{at least one collision}[= P[Y|C] = \frac{P[Y \cap C]}{P[C]} = \frac{0.0128}{0.0583} = 0.2196.$$

Answer D

24. R_1 , R_2 and R_3 denote the events that the 1st, 2nd and 3rd ball chosen is red, respectively.

$$P[R_3 \cap R_2 \cap R_1] = P[R_3|R_2 \cap R_1] \times P[R_2 \cap R_1]$$
$$= P[R_3|R_2 \cap R_1] \times P[R_2|R_1] \times P[R_1] = 1 \times \frac{5}{11} \times \frac{5}{10} = \frac{5}{22}.$$

Answer D

25. From the given information, 400 of those surveyed are both hockey and lacrosse fans, 200 are lacrosse fans and not hockey fans, and 400 are hockey fans and not lacrosse fans. This is true because there are 1000 fans in the survey, but a combined total of 800 + 600 = 1400 sports preferences, so that 400 must be fans of both. Of the 600 lacrosse fans, 400 are also hockey fans, so 200 are not hockey fans. The probability that a Canadian sports fans is not a hockey fan given that she/he is a lacrosse fan is $\frac{200}{600} = \frac{1}{3}$.

Answer C

26. Suppose there are B blue balls in urn II.

$P[\text{both balls are same color}] = P[\text{both blue} \cup \text{both red}] = P[\text{both blue}] + P[\text{both red}]$

(the last equality is true since the events "both blue" and "both red" are disjoint).

$P[\text{both blue}] = P[\text{blue from urn I} \cap \text{blue from urn II}]$

$= P[\text{blue from urn I}] \times P[\text{blue from urn II}]$ (choices from the two urns are independent)

$= \frac{6}{10} \times \frac{B}{16+B}$,

$P[\text{both red}] = P[\text{rd from urn I} \cap \text{red from urn II}]$

$= P[\text{red from urn I}] \times P[\text{red from urn II}] = \frac{4}{10} \times \frac{16}{16+B}$,

We are given $\frac{6}{10} \times \frac{B}{16+B} + \frac{4}{10} \times \frac{16}{16+B} = 0.44 \rightarrow \frac{6B+64}{10(16+B)} = 0.44 \rightarrow B = 4$.

Answer A

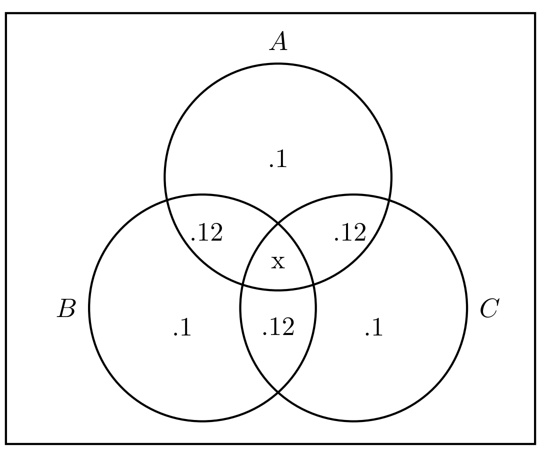
27. We are given $P[A \cap B' \cap C'] = P[A' \cap B \cap C'] = P[A' \cap B' \cap C] = 0.1$ (having exactly one risk factor means not having either of the other two). We are also given

$P[A \cap B \cap C'] = P[A \cap B' \cap C] = P[A' \cap B \cap C] = 0.12$.

And we are given $P[A \cap B \cap C | A \cap B] = \frac{1}{3}$. We are asked to find $P[A' \cap B' \cap C' | A']$.

From $P[A \cap B \cap C | A \cap B] = \frac{1}{3}$ we get $\frac{P[A \cap B \cap C]}{P[A \cap B]} = \frac{1}{3}$, and then $P[A \cap B \cap C] = \frac{1}{3} \times P[A \cap B]$.

The following Venn diagram illustrates the situation:



We see that $P[A \cap B \cap C] = x$ and $P[A \cap B] = x + 0.12$, so that

$x = \frac{1}{3} \times (x + 0.12) \ \rightarrow \ x = P[A \cap B \cap C] = 0.06$.

Alternatively, we can use the rule $P[D] = P[D \cap E] + P[D \cap E']$ to get

$P[A \cap B] = P[A \cap B \cap C] + P[A \cap B \cap C'] = P[A \cap B \cap C] + 0.12$.

Then, $P[A \cap B] = P[A \cap B \cap C] + 0.12 = \frac{1}{3} \times P[A \cap B] + 0.12 \ \rightarrow \ P[A \cap B] = 0.18$

and $P[A \cap B \cap C] = \frac{1}{3} \times 0.18 = 0.06$.

We can also see from the diagram that $P[A \cap B'] = 0.1 + 0.12 = 0.22$.

Alternatively, we can use the rule above again to get

$P[A \cap B'] = P[A \cap B' \cap C] + P[A \cap B' \cap C'] = 0.12 + 0.1 = 0.22$.

Then, $P[A] = P[A \cap B] + P[A \cap B'] = 0.18 + 0.22 = 0.4$, and $P[A'] = 1 - P[A] = 0.6$.

We are asked to find $P[A' \cap B' \cap C' | A'] = \frac{P[A' \cap B' \cap C']}{P[A']} = \frac{P[A' \cap B' \cap C']}{0.6}$, so we must find

$P[A' \cap B' \cap C']$. From the Venn diagram, we see that

$P[A' \cap B' \cap C'] = 1 - (0.1 + 0.1 + 0.1 + 0.12 + 0.12 + 0.12 + 0.06) = 0.28$.

Finally, $P[A' \cap B' \cap C' | A'] = \frac{P[A' \cap B' \cap C']}{P[A']} = \frac{P[A' \cap B' \cap C']}{0.6} = \frac{0.28}{0.6} = 0.467$.

Answer C

PRACTICE EXAMS

Note that some of the questions on these practice exams are somewhat more challenging than the typical exam questions.

PRACTICE EXAM 1

1. If E and F are events for which $P[E \cup F] = 1$, then $P[E' \cup F'] =$

(A) 0
(B) $P[E'] + P[F'] - P[E'] \times P[F']$
(C) $P[E'] + P[F']$
(D) $P[E'] + P[F'] - 1$
(E) 1

2. Sixty percent of new drivers have had driver education. During their first year, new drivers without driver education have a probability of 0.08 of having an accident, but new drivers with driver education have only a 0.05 probability of an accident. What is the probability a new driver has had driver education, given that the driver has had no accidents the first year?

(A) $\dfrac{5}{6}$
(B) $\dfrac{0.92 \times 0.4}{0.95 \times 0.6 + 0.92 \times 0.4}$
(C) $\dfrac{0.95 \times 0.4}{0.95 \times 0.6 + 0.92 \times 0.4}$
(D) $\dfrac{0.95 \times 0.4}{0.95 \times 0.4 + .92 \times 0.6}$
(E) $\dfrac{0.95 \times 0.6}{0.95 \times 0.6 + 0.92 \times 0.4}$

3. A loss distribution random variable X has a pdf of $f(x) = ae^{-x} + be^{-2x}$ for $x > 0$. If the mean of X is 1, find the probability $P[X < 1]$.

(A) 0.52 (B) 0.63 (C) 0.74 (D) 0.85 (E) 0.96

4. The number of claims made on an insurance policy in a year has a Poisson distribution with mean 3. The policy does not cover the first two claims but covers all claims after the first two. Determine the expected number of claims that the insurance policy will cover in one year.

(A) Less than 1.0
(B) At least 1.0 but less than 1.2
(C) At least 1.2 but less than 1.4
(D) At least 1.4 but less than 1.6
(E) At least 1.6

5. X has a binomial distribution with mean 1.5 and variance 0.375. Y has a discrete uniform distribution on the integers from 1 to 3. X and Y are independent. Determine $E[X^Y]$.

(A) 1.25 (B) 1.725 (C) 2.125 (D) 3.0 (E) 3.625

6. If X has a normal distribution with mean 1 and variance 4, then $P[X^2 - 2X \leq 8] =?$

(A) 0.13 (B) 0.43 (C) 0.75 (D) 0.86 (E) 0.93

7. The pdf of X is $f(x) = \begin{cases} 2x & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$. The mean of X is μ. Find $\dfrac{E[\,|X-\mu|\,]}{Var[X]}$.

(A) $\frac{20}{9}$ (B) $\frac{26}{9}$ (C) $\frac{32}{9}$ (D) $\frac{19}{81}$ (E) $\frac{22}{81}$

8. Two players put one dollar each into a pot. They decide to throw a pair of dice alternately. The first one who throws a total of 5 on both dice wins the pot. How much should the player who starts add to the pot to make this a fair game?

(A) $\frac{9}{17}$ (B) $\frac{8}{17}$ (C) $\frac{1}{8}$ (D) $\frac{2}{9}$ (E) $\frac{8}{9}$

9. An analysis of economic data shows that the annual income of a randomly chosen individual from country A has a mean of $18,000 and a standard deviation of $6000, and the annual income of a randomly chosen individual from country B has a mean of $31,000 and a standard deviation of $8000. 100 individuals are chosen at random from Country A and 100 from Country B. Find the approximate probability that the average annual income from the group chosen from Country B is at least $15,000 larger than the average annual income from the group chosen from Country A (all amounts are in US$).

(A) 0.9972 (B) 0.8413 (C) 0.5000 (D) 0.1587 (E) 0.0228

10. Three individuals are running a one kilometer race. The completion time for each individual is a random variable. X_i is the completion time, in minutes, for person i.

X_1 : uniform distribution on the interval $[2.9\,,3.1]$

X_2 : uniform distribution on the interval $[2.7\,,3.1]$

X_3 : uniform distribution on the interval $[2.9\,,3.3]$

The three completion times are independent of one another.

Find the expected latest completion time (nearest 0.1).

(A) 2.9 (B) 3.0 (C) 3.1 (D) 3.2 (E) 3.3

11. The amount of a claim paid by an insurer on a particular policy is a continuous random variable with pdf $f(x) = 4x^3$ for $0 < x < 1$. The insurer has three independent policies of this type. Determine the variance of the second largest of the three independent claims.

(A) 0.0120 (B) 0.0125 (C) 0.0130 (D) 0.0135 (E) 0.0140

12. A loss random variable X has a uniform distribution on the interval [0,1000].

Find the variance of the insurer payment per loss if there is a deductible of amount 100 and a policy limit (maximum insurance payment) of amount 400 (nearest 1000).

(A) 20,000 (B) 21,000 (C) 22,000 (D) 23,000 (E) 24,000

13. X has a lognormal distribution with parameters $\mu = 1$ and $\sigma = 1$. The probability $P[X \leq 1]$ is denoted a. The normal approximation (without integer correction) is applied to X, and the same probability is found based on the normal approximation and is denoted b. Find $b - a$.

(A) Less than 0
(B) At least 0 but less than 0.05
(C) At least 0.05 but less than 0.10
(D) At least 0.10 but less than 0.15
(E) At least 0.15

14. Let X be a random variable with mean 3 and variance 2, and let Y be a random variable such that for every x, the conditional distribution of Y given $X = x$ has a mean of x and a variance of x^2. What is the variance of the marginal distribution of Y?

(A) 2 (B) 4 (C) 5 (D) 11 (E) 13

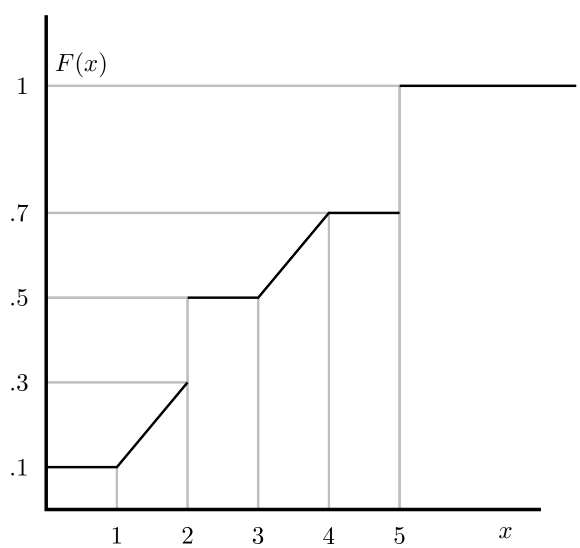
15. Let X and Y be discrete random variables with joint probabilities given by

Y \ X	1	5
2	$\theta_1 + \theta_2$	$\theta_1 + 2\theta_2$
4	$\theta_1 + 2\theta_2$	$\theta_1 + \theta_2$

Let the parameters θ_1 and θ_2 satisfy the usual assumption associated with a joint probability distribution and the additional constraints $-0.25 \leq \theta_1 \leq 0.25$ and $0 \leq \theta_2 \leq 0.35$. If X and Y are independent, then $(\theta_1, \theta_2) =$

(A) $\left(0, \frac{1}{6}\right)$ (B) $\left(\frac{1}{4}, 0\right)$ (C) $\left(-\frac{1}{4}, \frac{1}{3}\right)$ (D) $\left(-\frac{1}{8}, \frac{1}{4}\right)$ (E) $\left(\frac{1}{16}, \frac{1}{8}\right)$

16. Below is the graph of the cumulative distribution function for the random variable X:



Determine the probability $P[2 \leq X^2 \leq 5]$.

(A) 0.2379 (B) 0.3172 (C) 0.4758 (D) 0.5551 (E) 0.6344

17. For a Poisson random variable X with mean λ it is found that it is twice as likely for X to be less than 3 as it is for X to be greater than or equal to 3. Find λ (nearest .1).

(A) 2.0 (B) 2.2 (C) 2.4 (D) 2.6 (E) 2.8

18. Let X and Y be discrete random variables with joint probability function

$$f(x,y) = \begin{cases} \dfrac{2^{x+1-y}}{9} & \text{for } \ x = 1,2 \text{ and } y = 1,2, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $E\left[\dfrac{X}{Y}\right]$.

(A) $\dfrac{8}{9}$ (B) $\dfrac{5}{4}$ (C) $\dfrac{4}{3}$ (D) $\dfrac{25}{18}$ (E) $\dfrac{5}{3}$

19. People passing by a city intersection are asked for the month in which they were born. It is assumed that the population is uniformly divided by birth month, so that any randomly passing person has an equally likely chance of being born in any particular month. Find the minimum number of people needed so that the probability that no two people have the same birth month is less than 0.5.

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

20. Under a group insurance policy, an insurer agrees to pay 100% of the medical bills incurred during the year by employees of a small company, up to a maximum total of one million dollars. The total amount of bills incurred, X, has probability density function

$$f(x) = \begin{cases} \dfrac{x(4-x)}{9} & 0 < x < 3, \\ 0 & \text{otherwise,} \end{cases}.$$

where x is measured in millions. Calculate the total amount, in millions of dollars, the insurer would expect to pay under this policy.

(A) 0.120 (B) 0.301 (C) 0.935 (D) 2.338 (E) 3.495

21. X has a beta distribution with parameters a and b with pdf

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a) \times \Gamma(b)} \times x^{a-1} \times (1-x)^{b-1} \text{ for } 0 < x < 1.$$

Assuming that $b = 2a$, find the variance of X.

(A) $\dfrac{1}{3(9a+3)}$ (B) $\dfrac{2}{3(9a+3)}$ (C) $\dfrac{1}{3(3a+1)}$ (D) $\dfrac{4}{3(9a+3)}$ (E) $\dfrac{5}{3(9a+3)}$

22. A carnival gambling game involves spinning a wheel and then tossing a coin. The wheel lands on one of three colors, red, white or blue. There is a 1/2 chance that the wheel lands on red, and there is a 3/8 chance of white and a 1/8 chance of blue. A coin of the color indicated by the wheel is then tossed. Red coins have a 50% chance of tossing a head, white coins have a 3/4 chance of tossing a head, and blue coins have a 7/8 chance of tossing a head. If the game player tosses a head, she wins \$100, if she does not toss a head she wins 0. Find the cost to play the game so that the carnival wins an average of \$1 per play of the game.

(A) Less than 62
(B) At least 62 but less than 64
(C) At least 64 but less than 66
(D) At least 66 but less than 68
(E) At least 68

23. Fred, Ned and Ted each have season tickets to the Toronto Rock (Lacrosse). Each one of them might, or might not attend any particular game. The probabilities describing their attendance for any particular game are

$P[\text{at least one of them attends the game}] = 0.95,$

$P[\text{at least two of them attend the game}] = 0.80$ and

$P[\text{all three of them attend the game}] = 0.50$

Their attendance pattern is also symmetric in the following way

$$P[F] = P[N] = P[T] \text{ and } P[F \cap N] = P[F \cap T] = P[N \cap T]$$

where F, N and T denote the events that Fred, Ned and Ted attended the game, respectively. For a particular game, find the probability that Fred and Ned attended.

(A) .15 (B) .30 (C) .45 (D) .60 (E) .75

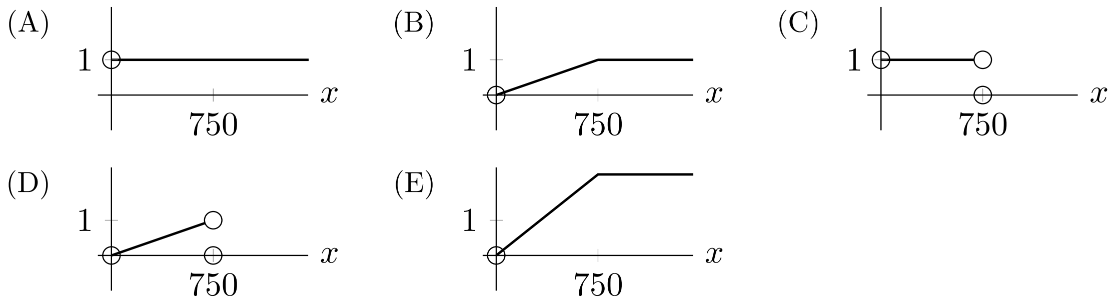
24. An insurer will pay the amount of a loss in excess of a deductible amount α. Suppose that the loss amount has a continuous uniform distribution between 0 and $C > \alpha$. When a loss occurs, let the expected payout on the policy be $f(\alpha)$. Find $f'(\alpha)$.

(A) $\frac{\alpha}{C}$ (B) $-\frac{\alpha}{C}$ (C) $\frac{\alpha}{C} + 1$ (D) $\frac{\alpha}{C} - 1$ (E) $1 - \frac{\alpha}{C}$

25. Coins K and L are weighted so the probabilities of heads are 0.3 and 0.1, respectively. Coin K is tossed 5 times and coin L is tossed 10 times. If all the tosses are independent, what is the probability that coin K will result in heads 3 times and coin L will result in heads 6 times?

(A) $\binom{5}{3}(0.3)^3(0.7)^2 + \binom{10}{6}(0.1)^3(0.9)^2$
(B) $\binom{5}{3}(0.3)^3(0.7)^2\binom{10}{6}(0.1)^6(0.9)^4$
(C) $\binom{15}{9}(0.4)^9(0.6)^6$
(D) $\frac{\binom{5}{3}\binom{10}{6}}{\binom{15}{9}}$
(E) 0.6×0.9

26. An insurance policy is written that reimburses the policyholder for all losses incurred up to a benefit limit of 750 . Let $f(x)$ be the benefit paid on a loss of x. Which of the following most closely resembles the graph of the derivative of f?



27. The value, v, of an appliance is based on the number of years since purchase, t, as follows:

$v(t) = e^{(7-0.2t)}$. If the appliance fails within seven years of purchase, a warranty pays the owner the value of the appliance. After seven years the warranty pays nothing. The time until failure of the appliance has an exponential distribution with a mean of 10. Calculate the expected payment from the warranty.

(A) 98.70 (B) 109.66 (C) 270.43 (D) 320.78 (E) 352.16

28. A test for a disease correctly diagnoses a diseased person as having the disease with probability 0.85. The test incorrectly diagnoses someone without the disease as having the disease with a probability of 0.10. If 1% of the people in a population have the disease, what is the chance that a person from this population who tests positive for the disease actually has the disease?

(A) 0.0085 (B) 0.0791 (C) 0.1075 (D) 0.1500 (E) 0.9000

29. Let X and Y be discrete random variables with joint probability function $f(x,y)$ given by the following table:

Y \ X	2	3	4	5
0	0.05	0.05	0.15	0.05
1	0.40	0	0	0
2	0.05	0.15	0.10	0

For this joint distribution, $E[X] = 2.85$ and $E[Y] = 1$. Calculate $Cov[X,Y]$.

(A) -0.20 (B) -0.15 (C) 0.95 (D) 2.70 (E) 2.85

30. One of the questions asked by an insurer on an application to purchase a life insurance policy is whether or not the applicant is a smoker. The insurer knows that the proportion of smokers in the general population is 0.30, and assumes that this represents the proportion of applicants who are smokers. The insurer has also obtained information regarding the honesty of applicants:

- 40% of applicants that are smokers say that they are non-smokers on their applications,
- none of the applicants who are non-smokers lie on their applications.

What proportion of applicants who say they are non-smokers are actually non-smokers?

(A) 0 (B) $\frac{6}{41}$ (C) $\frac{12}{41}$ (D) $\frac{35}{41}$ (E) 1

PRACTICE EXAM 1: SOLUTIONS

Answer Key

Question #	Answer
1	C
2	E
3	B
4	C
5	D
6	D
7	C
8	C
9	E
10	C
11	B
12	D
13	D
14	E
15	B

Question #	Answer
16	B
17	A
18	D
19	D
20	C
21	B
22	C
23	D
24	D
25	B
26	C
27	D
28	B
29	B
30	D

1. $P[E' \cup F'] = P[E'] + P[F'] - P[E' \cap F']$.
But $E' \cap F' = (E \cup F)'$, so that $P[E' \cap F'] = P[(E \cup F)'] = 1 - P[E \cup F] = 1 - 1 = 0$, so that $P[E' \cup F'] = P[E'] + P[F']$.

Answer C

2. We define the following events:

A - the new driver has had driver education

B - the new driver has had an accident in his first year.

We are to find $P[A|\overline{B}] = \dfrac{P[A \cap \overline{B}]}{P[\overline{B}]}$, and we are given $P[A] = .6\,, P[B|\overline{A}] = 0.08$, and $P[B|A] = 0.05$ .Using rules of probability, $P[\overline{B}|A] = 1 - P[B|A] = 0.95$, and hence, $P[A \cap \overline{B}] = P[\overline{B}|A] \times P[A] = 0.95 \times 0.6$.Also, $P[\overline{A}] = 1 - P[A] = 0.4$.

But, $P[\overline{B}|\overline{A}] = 1 - P[B|\overline{A}] = 1 - .08 = 0.92$, and hence

$P[\overline{A} \cap \overline{B}] = P[\overline{B}|\overline{A}] \times P[\overline{A}] = 0.92 \times 0.4$.

Thus, $P[\overline{B}] = P[A \cap \overline{B}] + P[\overline{A} \cap \overline{B}] = 0.95 \times 0.6 + .92 \times 0.4$, and then

$P[A|\overline{B}] = \dfrac{P[A \cap \overline{B}]}{P[\overline{B}]} = \dfrac{0.95 \times 0.6}{0.95 \times 0.6 + 0.92.4}$.

Answer E

3. $f(x) = ae^{-x} + be^{-2x} \;\rightarrow\; \int_0^\infty f(x)\,dx = a + \dfrac{1}{2}b = 1$.
We use the following integral rule for integer $k \geq 0$ and $c > 0$,$\int_0^\infty x^k\, e^{-cx}\,dx = \frac{k!}{c^{k+1}}$, to get

$$E[X] = \int_0^\infty x\,f(x)\,dx = a + \frac{1}{4}b = 1.$$

Solving the equations results in $a = 1\,, b = 0$. The probability is

$P[X < 1] = \int_0^1 e^{-x}\,dx = 1 - e^{-1} = 0.632$

Answer B

4. We denote the number of claims in a year by N, a Poisson random variable with mean 3. The number of claims covered by the insurer is $\max\{0, N-2\}$, which is the number of claims above 2. This is algebraically the same as a deductible of 2 applied to N. For any non-negative random variable X, when applying a deductible d to X it is always the case that $\max\{0, X-d\} = X - \min\{X,d\}$, so that $E[\max\{0, X-d\}] = E[X] - E[\min\{X,d\}]$. Note that $\min\{X,d\}$ is the amount paid on a loss of amount X up to a limit of d. It is the case for most probability distributions that it is easier to calculate $\min\{X,d\}$ than $\max\{0, N-2\}$. In this case $d = 2$ and

$$\min\{N,2\} = \begin{cases} 0 & N = 0 \quad \text{prob. } N = 0 \\ 1 & N = 1 \quad \text{prob. } N = 1 \\ 2 & N \geq 2 \quad \text{prob. } N \geq 2 \end{cases}.$$