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Study Manual for CAS Exam MAS-I

8th Edition

by

Ambrose Lo, PH.D., FSA, CERA

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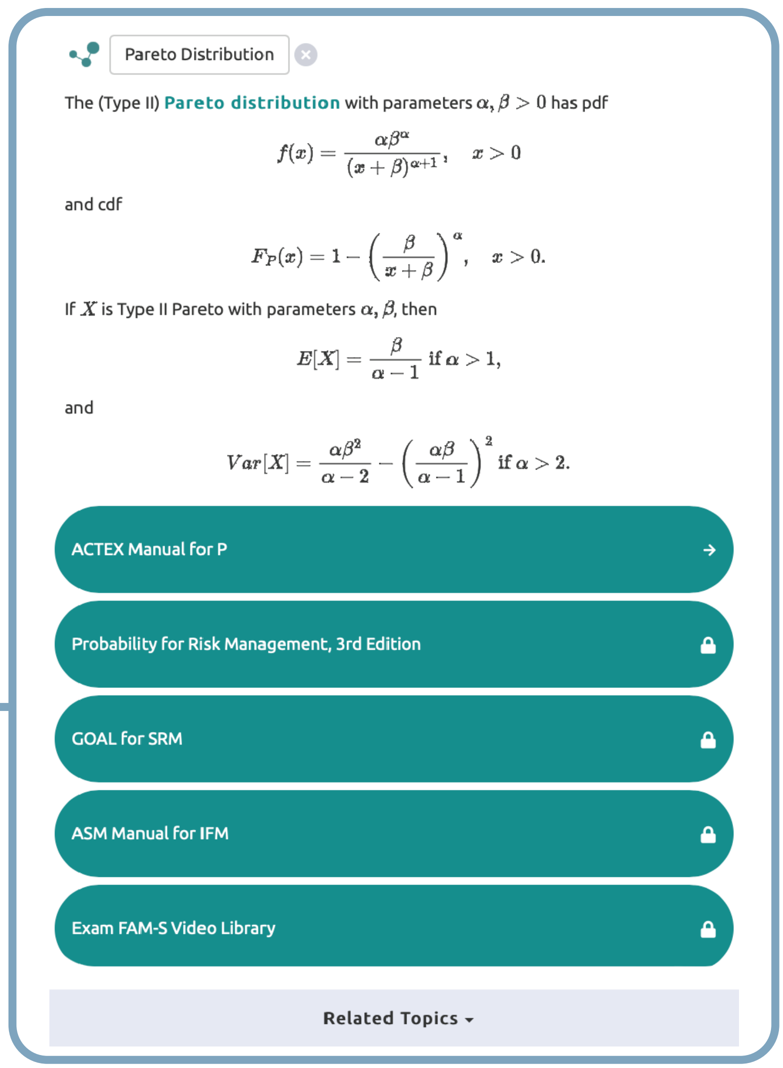

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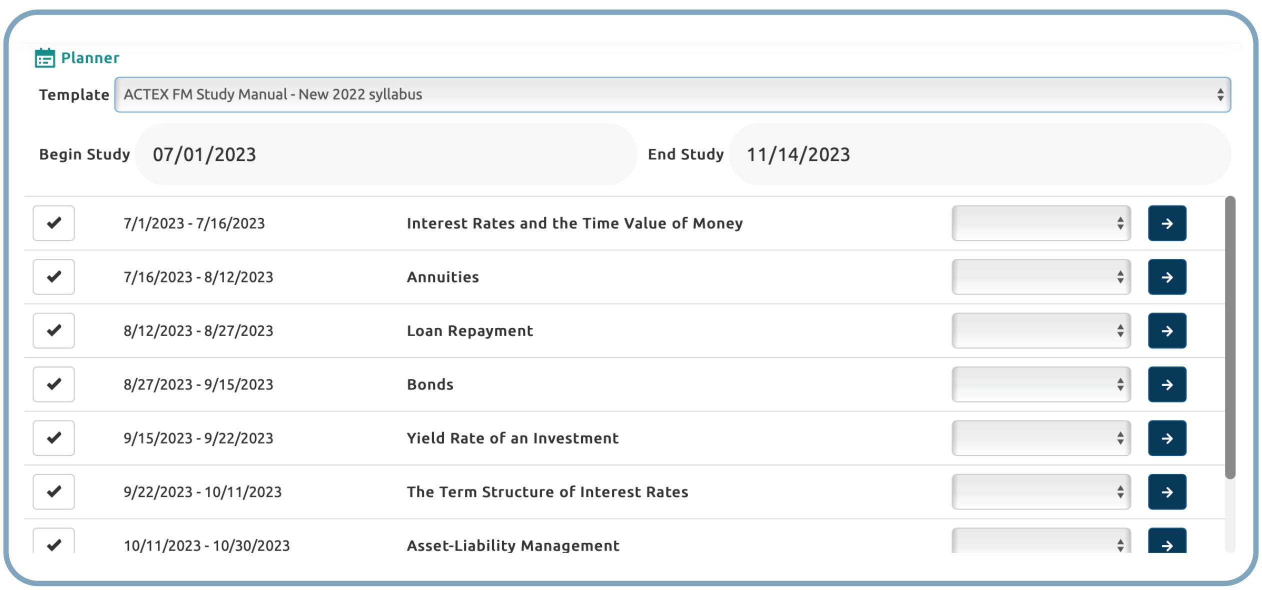

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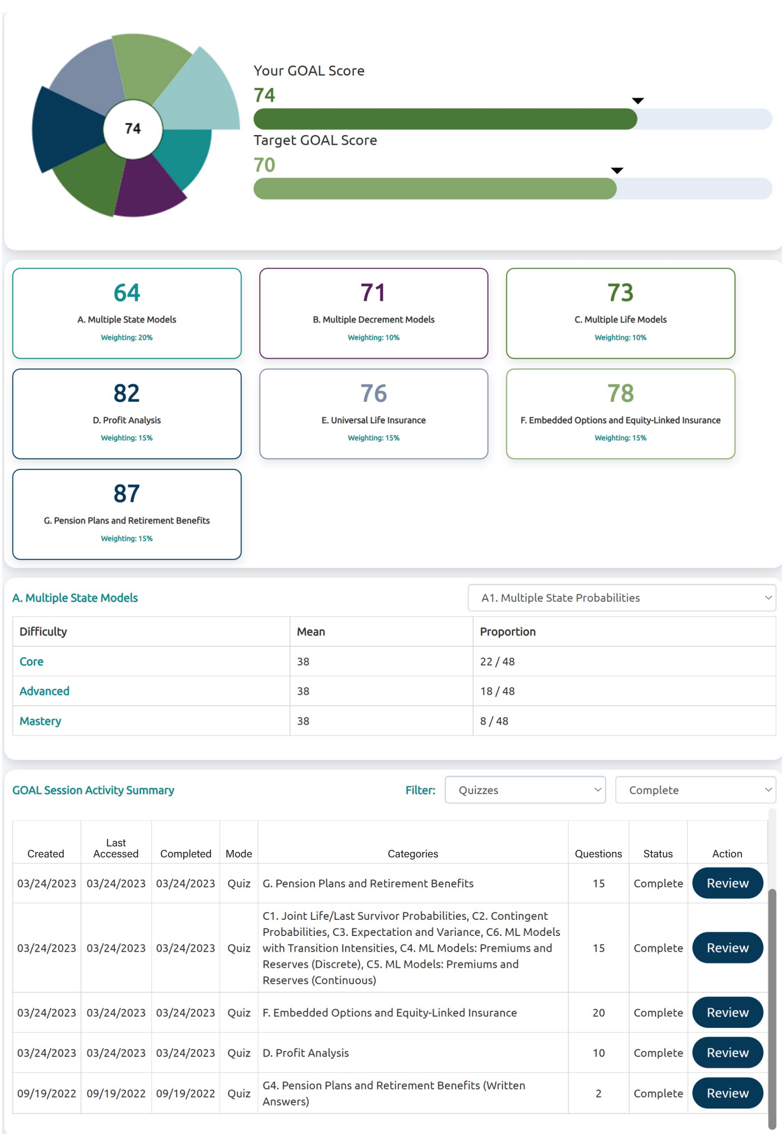

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Contents

Preface

NOTE TO STUDENTS

Please read this preface carefully. It contains very important information that will help you navigate this study manual and Exam MAS-I smoothly! 👍

1 About Exam MAS-I

Exam Theme

In 2018, the Casualty Actuarial Society (CAS) added a considerable amount of material on predictive analytics to its Associateship curriculum in view of the growing relevance of this discipline to actuarial work, especially in property and casualty (P&C) insurance. The most significant changes were the introduction of two exams bearing **"Modern Actuarial Statistics (MAS)"** in their title: **Exam MAS-I** and **Exam MAS-II**, both of which serve to enhance the predictive modeling and data science skill set of P&C actuaries in this day and age. This study manual prepares you adequately for Exam MAS-I, but also paves the way for Exam MAS-II.

As its name suggests, you will see a lot of "statistics" in Exam MAS-I, some classical and some "modern." Besides doing probability calculations and classical statistical inference (e.g., parameter estimation and hypothesis testing), you will also work with statistical models and study some contemporary predictive modeling techniques. You will learn the general tools available for constructing and evaluating predictive models (e.g., training/test set split, cross-validation), and the technical details of specific types of models (e.g., linear models, generalized linear models, shrinkage methods). After taking (and, with the use of this study manual, passing!) MAS-I, you will gain the foundational knowledge behind the modeling process.

According to the current syllabus available from

https://www.casact.org/exam/exam-mas-i-modern-actuarial-statistics-i,

the exam consists of three domains (or sections), covering miscellaneous topics in applied probability, mathematical statistics, and statistical modeling:

(As a conservative estimate, you will need *four months* of intensive study to master the material in this exam.)

Domain	Domain Weight
A. Probability Models (Stochastic Processes and Survival Models)	20-30%
B. Statistics	20-30%
C. Extended Linear Models [**MOST IMPORTANT!**]	45-55%

As you can see, each of Domains A and B accounts for about 25% of the exam, and Domain C alone occupies about 50%.

(The exam syllabus used to have Domain D: Time Series with Constant Variance, but it has been moved to Exam MAS-II effective from Fall 2023. MAS-I students now have less to study...a bit less!)

Exam Style

With effect from Fall 2023, Exam MAS-I is a 4-hour exam with a 15-minute scheduled break (the 15 minutes will be separate from the 4-hour exam window). It will feature the following item types: (Previously, it was a purely multiple-choice exam.)

- *Multiple Choice*

  Multiple answer choices are presented after a problem with only one correct answer. Traditionally, multiple-choice questions have five answer choices, most of which are in the form of ranges, e.g.:

  A. Less than 1%
  B. At least 1%, but less than 2%
  C. At least 2%, but less than 3%
  D. At least 3%, but less than 4%
  E. At least 4%

  If your answer is much lower than the bound indicated by Answer A or much higher than that suggested by Answer E, do check your calculations. The chances are that you have made computational mistakes, but this is not definitely the case (sometimes the CAS examiners themselves made a mistake!).

- *(Partially new) Multiple Selection*

  Multiple answer choices are presented after a problem with more than one correct answer. You will have to select all choices that apply.

- *(New!) Point and Click*

  An image is presented after a problem where the candidate must identify the correct area of the image by clicking on the correct location in the image.

- *(New!) Fill in the Blank*

  A blank section is presented after a problem where the candidate must input the correct value.

The CAS has yet to release sample questions for the new item types. Regardless of the types of question, you will learn the exam material in essentially the same way, paying attention to:

- Mathematical formulas and their applications, in order to gain computational proficiency

  (This is the traditional way you approach an actuarial exam.)

- **[IMPORTANT!]** The conceptual aspects of various statistical techniques in the syllabus

  (Unlike P and FM, conceptual issues will figure prominently in MAS-I, especially in Domain C; see Chapter 11. For MAS-I, you will need to spend time understanding the nitty-gritty details of different types of predictive model.)

Historical Pass Ratios

Based on the exam statistics available from

https://www.casact.org/exams-admissions/exams-results-summary-exam-statistics,

the table below shows the number of candidates, pass ratios, and effective pass ratios for Exam MAS-I since it was offered in Spring 2018.

Sitting	# Candidates	# Passing Candidates	Pass Ratio	Effective Pass Ratio
Spring 2023	(TBA)	(TBA)	(TBA)	(TBA)
Fall 2022	845	440	52.1%	55.1%
Spring 2022	816	415	50.8%	64.8%
Fall 2021	791	437	55.2%	62.4%
Spring 2021	866	371	42.8%	50.3%
Fall 2020	1224	541	44.2%	52.3%
Spring 2020	(exam cancelled due to COVID)			
Fall 2019	992	398	40.1%	47.3%
Spring 2019	948	295	31.1%	36.4%
Fall 2018	846	259	30.6%	36.4%
Spring 2018	499	229	45.9%	50.3%

The pass ratios of MAS-I are typically in the 40-50% range, reflecting on the difficulty of this exam compared to other Associateship-level exams you have taken. It remains to be seen whether the pass ratio will go up or down after the new exam format takes effect in Fall 2023.

Exam Tables

In the exam, you will be supplied with a variety of tables, including:

- *Standard normal distribution table* (used throughout Parts I and II of this study manual)

  You will need this table for values of the standard normal distribution function or standard normal quantiles, when you work with normally distributed random variables or perform normal approximation.

- *Illustrative Life Table* (used mostly in Chapter 4 of this study manual)

  You will need this table when you are told that mortality of the underlying population follows the Illustrative Life Table.

- *A table of distributions for a number of common continuous and discrete distributions and the formulas for their moments and other probabilistic quantities* (used mostly in Part II of this study manual)

  This big table provides a great deal of information about some common (e.g., exponential, gamma, lognormal, Pareto) as well as less common distributions (e.g., inverse exponential, inverse Gaussian, Pareto, Burr, etc.). When an exam question centers on these distributions and quantities such as their means or variances are needed, consult this table.

- *Quantiles of t-distribution, F-distribution, chi-square distribution* (used in Chapters 8, 10, 12 and 13 of this study manual)

  These quantiles will be of use when you perform parametric hypothesis tests.

I strongly encourage you to download these tables from

https://www.casact.org/sites/default/files/2021-03/masi_tables.pdf

right away, print out a copy, and learn how to locate the relevant entries as you work out the examples and problems in this manual.

2 About this Study Manual

What is Special about This Study Manual?

I fully understand that you have an acutely limited amount of study time and that the MAS-I exam syllabus is insanely broad. With this in mind, the overriding objective of this study manual is to help you grasp the material in Exam MAS-I as effectively and efficiently as possible, so that you will pass the exam on your first try with ease and go on to Exam MAS-II with confidence. Here are some unique features of this manual to make this possible:

- Each chapter or section starts by explicitly stating which learning objectives and outcomes of the MAS-I exam syllabus we are going to cover, to assure you that we are on track and hitting the right target.

- The explanations in each chapter are thorough, but exam-focused and integrated with carefully chosen past exam/sample questions for illustration, so that you will learn the syllabus material effectively and efficiently. Throughout, I strive to keep you motivated by showing you how different concepts are typically tested, how different formulas are used, and where the exam focus lies in each section. As you read, you will develop a solid understanding of the concepts in MAS-I and know how to study for the exam.

- Formulas and results of utmost importance are $\boxed{\text{boxed}}$ for easy identification and numbered (in the (X.X.X) format) for later references. Mnemonics and shortcuts are emphasized, so are highlights of important exam items and common mistakes committed by students.

- While the focus of this study manual is on exam preparation, I take every opportunity to explain the intuitive meaning and mathematical structure of various formulas in the syllabus. The interpretations and insights you see will foster a genuine understanding of the syllabus material and reduce the need for slavish memorization. It is my belief and personal experience that a solid understanding of the underlying concepts is always conducive to achieving good exam results.

- To succeed in any actuarial exam, I can't overemphasize the importance of practicing a wide variety of exam-type problems to sharpen your understanding and develop proficiency. This study manual embraces this learning by doing approach and intersperses its expositions with approximately **600 in-text examples** and **900 end-of-chapter/section problems** (the harder ones are labeled as **[HARDER!]** or **[VERY HARD!!]**), which are either taken/adapted from relevant SOA/CAS past exams or original, all with step-by-step solutions, to consolidate your understanding and give you a sense of what you can expect to see in the real exam. As a general guide, you should:

  - ▷ Study *all* of the in-text examples, paying particular attention to recent CAS past exam questions.

  - ▷ Work out at least *half* of the end-of-chapter/section problems. Of course, the more problems you do, the better.

- Three full-length practice exams designed to mimic the real MAS-I exam in terms of style and difficulty conclude this study manual and give you a holistic review of the syllabus material. Detailed illustrative solutions are provided.

> **NOTE**
>
> The three practice exams will be released after the CAS posts the sample questions for the new item types. They are expected to be available on www.actuarialuniversity.com in mid-September.

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- For **technical** issues (e.g., not able to access the manual on Actuarial University, extending your digital license, upgrading your product, exercising the Pass Guarantee), please email Customer Service at `support@actexlearning.com`.

- Questions related to **specific contents** of this manual, including potential errors (typographical or otherwise), can be directed to me (Ambrose) by emailing `ambrose-lo@uiowa.edu`.

> **NOTE**
>
> For a faster turnaround, it would be greatly appreciated if you could reach out to the appropriate email address. ☺

About the Author

Professor **Ambrose Lo**, PhD, FSA, CERA, is currently Associate Professor of Actuarial Science with tenure at the Department of Statistics and Actuarial Science, The University of Iowa. He earned his B.S. in Actuarial Science (first class honors) and PhD in Actuarial Science from The University of Hong Kong in 2010 and 2014, respectively, and attained his Fellowship of the Society of Actuaries (FSA) in 2013. He joined The University of Iowa as Assistant Professor of Actuarial Science in August 2014, and was tenured and promoted to Associate Professor in July 2019. His research interests lie in dependence structures, quantitative risk management as well as optimal (re)insurance. His research papers have been published in top-tier actuarial journals, such as *ASTIN Bulletin: The Journal of the International Actuarial Association*, *Insurance: Mathematics and Economics*, and *Scandinavian Actuarial Journal*.

Besides dedicating himself to actuarial research, Ambrose attaches equal importance to teaching and education, through which he nurtures the next generation of actuaries and serves the actuarial profession. He has taught courses on financial derivatives, mathematical finance, life contingencies, and statistics for risk modeling. He has authored or coauthored the *ACTEX Study Manuals for Exams MAS-I, MAS-II, PA, and SRM*, a *Study Manual for Exam FAM*, and the textbook *Derivative Pricing: A Problem-Based Primer* (2018) published by Chapman & Hall/CRC Press. Although helping students pass actuarial exams is an important goal of his teaching, inculcating students with a thorough understanding of the subject and concrete problem-solving skills is always his top priority. In recognition of his exemplary teaching, Ambrose has received a number of awards and honors ever since he was a graduate student, including the 2012 Excellent Teaching Assistant Award from the Faculty of Science, The University of Hong Kong, public recognition in the *Daily Iowan* as a faculty member "making a positive difference in students' lives during their time at The University of Iowa" for eight years in a row (2016 to 2023), and the 2019-2020 Collegiate Teaching Award from the College of Liberal Arts and Sciences, The University of Iowa.

Part I

Probability Models
(Stochastic Processes & Survival Models)

Chapter 2

Reliability Theory

LEARNING OBJECTIVES

5. Given the joint distribution of more than one source of failure in a system (or life) and using Poisson Process assumptions:

    - Calculate probabilities and moments associated with functions of these random variables' variances.
    - Understand difference between a series system (joint life) and parallel system (last survivor) when calculating expected time to failure or probability of failure by a certain time
    - Understand the effect of multiple sources of failure (multiple decrement) on expected system time to failure (expected lifetime)

Range of weight: 2-8 percent

Chapter overview: Reliability theory is mainly concerned with analyzing the distribution of the random lifetime of a multi-component system. In particular, we are interested in the probability that such a system will function at a certain point of time. This probability is termed the reliability of the system. In the first two sections of this relatively short chapter, we look at a system from a static point of view, focusing on whether the system operates at a particular point of time from the knowledge of which components are operating and on computing the reliability of the system. Several popular designs to arrange the components of a system are presented, and the resulting influence on the reliability of a system is examined. In Section 2.3, we prescribe probability distributions on the individual components of a system and investigate the overall lifetime distribution of the system dynamically over time.

2.1 Typical Systems

MAS-I KNOWLEDGE STATEMENT(S)

c. Time until failure of the system (life)

d. Time until failure of the system (life) from a specific cause

e. Time until failure of the system (life) for parallel or series systems with multiple components

f. Paths that lead to parallel or series system failure for systems with multiple components

g. Relationship between failure time and minimal path and minimal cut sets

h. Bridge system and defining path to failure

OPTIONAL SYLLABUS READING(S)

Ross, Sections 9.1 and 9.2

We start by introducing several common system designs, which permeate much of the whole chapter.

Series systems. A *series system* functions if and only if *all* of its components function. It fails as soon as one of the components fails. To express this fact mathematically, we define the binary[i] variable

$$x_i = \begin{cases} 1, & \text{if the } i\text{th component is functioning,} \\ 0, & \text{if the } i\text{th component has failed,} \end{cases}$$

to be the *state* of the i^{th} component and designate the vector $\mathbf{x} = (x_1, \ldots, x_n)$ representing which components are functioning as the *state vector* of the system. Corresponding to a state vector $\mathbf{x}$, the *structure function* of the system defined as

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functiong when the state vector is } \mathbf{x}, \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x}, \end{cases}$$

indicates whether the system as a whole is functioning. It is a convenient device which characterizes a system in the sense that different systems are distinguished by different structure functions ϕ.

For a series system, the structure function is given by

$$\boxed{\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \begin{cases} 1, & \text{if every component is functioning, i.e., } x_i = 1 \text{ for all } i, \\ 0, & \text{if any component has failed, i.e., } x_i = 0 \text{ for some } i. \end{cases}}$$

It may help to understand the meaning of a series system by representing it pictorially in Figure 2.1.1, where a signal initiated at the left must pass through each and every component for reception on the right (i.e., for the system to work).

[i] A binary variable takes the values of 0 and 1 only.

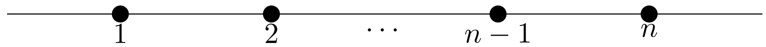


Figure 2.1.1: Pictorial representation of an n-component series system.

Parallel systems. In contrast to a series system, a *parallel system* functions if and only if *at least one* of its components is functioning. The corresponding structure function is

$$\begin{aligned} \phi(\mathbf{x}) &= \boxed{\max(x_1, \ldots, x_n)} \\ &= \begin{cases} 1, & \text{if at least one component is functioning, i.e., } x_i = 1 \text{ for some } i, \\ 0, & \text{if all components have failed, i.e., } x_i = 0 \text{ for all } i. \end{cases} \end{aligned}$$

A pictorial illustration of a parallel system is given in Figure 2.1.2.

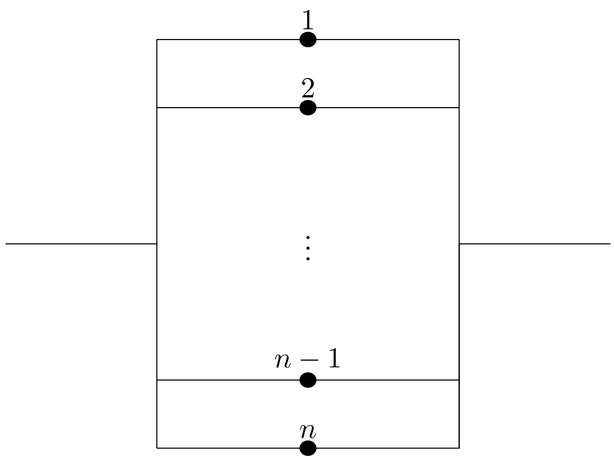


Figure 2.1.2: Pictorial representation of an n-component parallel system.

Other systems. Series and parallel systems form the basic building blocks of more complex systems. Here are some examples:

1. *k-out-of-n systems* (Example 9.3 of Ross): A *k-out-of-n system* with $1 \le k \le n$ is a generalization of series and parallel systems. It functions if and only if at least k out of the n components are functioning. The corresponding structure function is

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \ge k, \\ 0, & \text{if } \sum_{i=1}^n x_i < k. \end{cases}$$

   In particular, an n-out-of-n system is a series system (the functioning of all of the n components is required) and a 1-out-of-n system becomes a parallel system (as long as one component works, so does the whole system).

2. *A four-component hybrid system* (Example 9.4 of Ross): A system can consist of components which are themselves series, parallel, or k-out-of-n systems. As an illustration, consider the four-component system described in Figure 2.1.3. Observe that components 3 and 4 constitute a parallel system, which, together with components 1 and 2, forms a

series system—the system works if and only if both component 1 and component 2 ("series"), and at least one of components 3 and 4 work ("parallel"). The structure function of this hybrid system is given by

$$\phi(\mathbf{x}) = \boxed{\overbrace{x_1x_2 \times \underbrace{\max(x_3, x_4)}_{\text{parallel}}}^{\text{series}}},$$

which, along with the use of the useful identity

$$\boxed{\max(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i)} \tag{2.1.1}$$

for binary variables $x_1, \ldots, x_n$, can be further written as

$$\phi(\mathbf{x}) = x_1x_2[1 - (1 - x_3)(1 - x_4)] = \boxed{x_1x_2(x_3 + x_4 - x_3x_4)}.$$

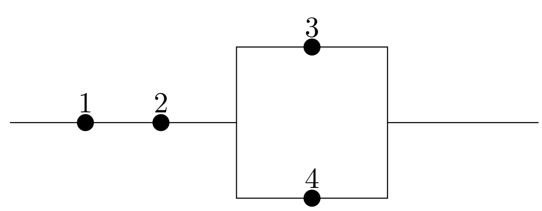


Figure 2.1.3: Pictorial representation of a four-component hybrid system.

3. *Bridge system* (Example 9.8 of Ross): A special type of systems explicitly stated in the knowledge statements of the exam syllabus is known as a bridge system, which is depicted in Figure 2.1.4. You may think of the path passing through component 3 in the middle as a "bridge."

   Expressing a bridge in terms of series and parallel structures by inspection is not an easy task. This will be achieved with the aid of minimal path and cut sets to be introduced in the next paragraph.

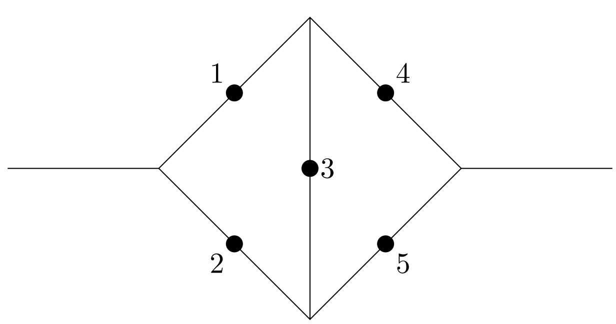


Figure 2.1.4: Pictorial representation of a bridge system.

[HARDER!] Minimal path and minimal cut sets. A deeper understanding of a system can be acquired by studying which critical components are sufficient for the system to work, and failure of which components alone will jeopardize the functioning of the system. These are formalized by the concepts of minimal path and minimal cut sets, which enable us to express an arbitrary system as a parallel representation of series structures, and as a series representation of parallel structures.

- *Minimal path sets:* A *minimal*[ii] *path set* is a set of components satisfying both properties below:

  1. *("Path")* The functioning of *each and every* component in this set guarantees the functioning of the whole system.
  2. *("Minimal")* The same conclusion in Point 1 cannot be drawn if you take away any component from the set.

  The above definition of a minimal path set is a bit delicate. Note that:

  ▷ Minimal path set is not unique. For an n-component parallel system, the sets $\{1\}$, $\{2\}, \ldots, \{n-1\}, \{n\}$ are all minimal path sets; each component alone constitutes a minimal path set. More generally,

  $$\boxed{\text{for a } k\text{-out-of-}n \text{ system, there are } \binom{n}{k} \text{ minimal path sets,}}$$

  namely, all of the sets comprising exactly k components; this is the content of Example 9.6 of Ross. This is because any k components suffice to ensure that the whole system functions and there are $\binom{n}{k}$ such combinations. Setting $k = 1$ and $k = n$, we get $\binom{n}{1} = n$ minimal path sets for an n-component parallel system and $\binom{n}{n} = 1$ minimal path set for an n-component series system.

  ▷ Different minimal path sets may also have different number of components. To see this, consider the bridge system depicted in Figure 2.1.4, for which the minimal path sets include $\{1,4\}, \{1,3,5\}, \{2,5\}$, and $\{2,3,4\}$. Some minimal path sets have two components while some have three.

  ▷ The fact that the system functions does not require each component in *all* minimal path sets to work. To put it in another way, the fact that a component in the set fails does *not* mean that the whole system fails! In the bridge system example, it suffices for the system to function with components 1 to 4 operating only, but with components 2 and 5 both failed (although $\{2,5\}$ is also a minimal path set).

  By the definition of minimal path sets, a system will function if and only if there is *at least one* minimal path set in which *all* components work. Notice the important order of the phrases "at least one" and "all". Recalling that "at least one" suggests a parallel structure while "all" means a series structure, we manage to express the structure function of any

[ii] In mathematics, there is a distinction between "minimal" and "minimum". We shall not pursue the subtle but substantive difference here.

system as a parallel representation of series systems:

$$\boxed{\phi(\mathbf{x}) = \underbrace{\max_j}_{\text{parallel}} \underbrace{\prod_{i\in A_j}}_{\text{series}} x_i,} \tag{2.1.2}$$

where the A_j's are the minimal path sets of the system. To check that (2.1.2) is true, we note the following equivalences:

$$\begin{aligned} \phi(\mathbf{x}) = 1 \quad &\Leftrightarrow \quad \text{there exists a minimal path set } A_j \text{ such that } \prod_{i\in A_j} x_i = 1 \\ &\Leftrightarrow \quad \text{there exists a minimal path set } A_j \text{ such that } x_i = 1 \text{ for all } i \in A_j \end{aligned}$$

The last statement is simply the definition of a minimal path set.

Example 2.1.1. (Based on Example 9.5 of Ross: Identification of minimal path set for a hybrid system) Consider the five-component system in Figure 2.1.5.

Determine the number of minimal path sets of the system.

A. 1

B. 2

C. 3

D. 4

E. 5

Solution. By inspection, there are four minimal path sets: $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 5\}$ and $\{2, 5\}$. **(Answer: D)** □

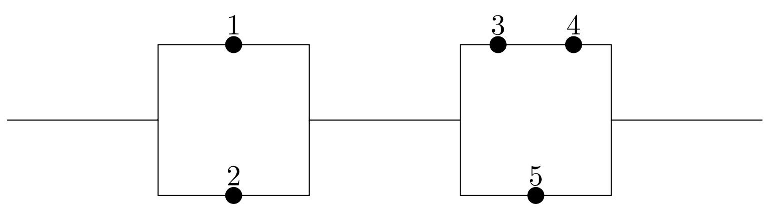


Figure 2.1.5: Pictorial representation of the five-component system in Example 2.1.1.

The next two Exam MAS-I/S past exam problems both center on the minimal path sets of k-out-of-n systems (series and parallel systems, in particular).

Example 2.1.2. (CAS Exam S Fall 2017 Question 8: Two k-out-of-n systems connected in parallel) A 3-out-of-50 system is placed in parallel with a 48-out-of-50-system to form a combined system.

Calculate the number of minimal path sets for the combined system.

A. Fewer than 20,000

B. At least 20,000, but fewer than 30,000

C. At least 30,000, but fewer than 40,000

D. At least 40,000, but fewer than 50,000

E. At least 50,000

Solution. The 3-out-of-50 system and 48-out-of-50 have $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets, respectively. When the two systems are put in parallel, each minimal path set of any of the two systems provides a minimal path set of the aggregate system. The number of minimal path sets increases to $19,600 + 1,225 = \boxed{20,825}$. **(Answer: B)** □

Example 2.1.3. (CAS Exam MAS-I Fall 2018 Question 7: Two k-out-of-n systems connected in series) A 3-out-of-50 system is placed in series with a 48-out-of-50-system.

Calculate the number of minimal path sets.

A. Fewer than 20,000

B. At least 20,000 but fewer than 100,000

C. At least 200,000 but fewer than 2,000,000

D. At least 2,000,000 but fewer than 20,000,000

E. At least 20,000,000

Ambrose's comments: This MAS-I problem is an adaptation of the preceding Exam S problem, with "placed in parallel" replaced by "placed in series."

Solution. Following the solution of the preceding example, the 3-out-of-50 system and 48-out-of-50-system have, respectively, $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets. When the two systems are put in series, a minimal path set of the aggregate system comprises a minimal path set from the 3-out-of-50 system and another minimal path set from the 48-out-of-50-system. Overall, the number of minimal path sets is $19,600 \times 1,225 = \boxed{24,010,000}$. **(Answer: E)** □

Remark. Answer B is probably intended to be "At least 20,000 but fewer than <u>2</u>00,000".

- *Minimal cut sets:* As a concept dual to a minimal path set, a *minimal cut set* consists of a set of components such that:

  1. *("Cut")* The failure of *all* components in the set guarantees the failure of the whole system.
  2. *("Minimal")* The same conclusion cannot be drawn if you take away any component from the set.

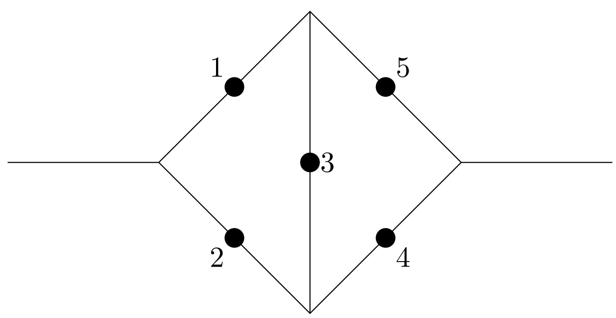
  By the definition of a minimal cut set, a system will fail if and only if there is *at least one* minimal cut set in which *all* components fail. This translates into the following *series* arrangement of parallel systems:

$$\boxed{\phi(\mathbf{x}) = \underbrace{\prod_j}_{\text{series}} \underbrace{\max_{i \in C_j}}_{\text{parallel}} x_i,} \tag{2.1.3}$$

  where the C_j's are the minimal cut sets. As a check, note the following equivalences:

$$\begin{aligned} \phi(\mathbf{x}) = 0 \quad &\Leftrightarrow \quad \text{there exists a minimal cut set } C_j \text{ such that } \max_{i \in C_j} x_i = 0 \\ &\Leftrightarrow \quad \text{there exists a minimal cut set } C_j \text{ such that } x_i = 0 \text{ for all } i \in C_j \end{aligned}$$

Example 2.1.4. (CAS Exam MAS-I Spring 2018 Question 9: Minimal path and cut sets of a bridge system) You are given the following system:



and the following statements:

I. $\{1,5\}$ and $\{2,4\}$ are minimal path sets.

II. $\{1,2\}$ and $\{4,5\}$ are minimal cut sets.

III. $\{1,3,5\}$ and $\{2,3,4\}$ are both minimal path sets and minimal cut sets.

Determine which of the above statements are correct.

A. None are correct

B. I and II only

C. I and III only

D. II and III only

E. The answer is not given by (A), (B), (C), or (D)

Solution. I&II. Correct, by inspection.

III. Incorrect. Because $\{1, 5\}$ is a minimal path set, $\{1, 3, 5\}$, with component 3 added, cannot serve as a minimal path set. The same idea applies to $\{2, 3, 4\}$. **(Answer: B)**

□

Remark. Statement III would be true if it says "$\{1, 3, 4\}$ and $\{2, 3, 5\}$ are both minimal path sets and minimal cut sets."

Example 2.1.5. **(Based on Example 9.8 of Ross: Manipulating the structure function of a bridge system)** Determine which of the following representation of the structure function of the bridge system in Figure 2.1.4 is correct.

A. $x_3 \max(x_1, x_2) \max(x_4, x_5)$

B. $x_3 \max(x_1x_4, x_2x_5)$

C. $(1 - x_3) \max(x_1, x_4) \max(x_2, x_5)$

D. $(1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_2x_5)(1 - x_2x_3x_4)$

E. $1 - (1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_2x_5)(1 - x_2x_3x_4)$

Solution. Since the minimal path sets of the bridge system are $\{1, 4\}, \{1, 3, 5\}, \{2, 5\}$ and $\{2, 3, 4\}$, using the parallel representation of series systems given in (2.1.2) we have

$$\max(x_1x_4, x_1x_3x_5, x_2x_5, x_2x_3x_4) = 1 - (1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_2x_5)(1 - x_2x_3x_4).$$

Thus **Answer E** is correct. □

Remark. As the minimal cut sets of the bridge system are $\{1, 2\}, \{1, 3, 5\}, \{2, 3, 4\}$ and $\{4, 5\}$, we can also use (2.1.3) to express the structure function as

$$\begin{aligned} & \max(x_1, x_2) \max(x_1, x_3, x_5) \max(x_2, x_3, x_4) \max(x_4, x_5) \\ = & [1 - (1 - x_1)(1 - x_2)][1 - (1 - x_1)(1 - x_3)(1 - x_5)] \\ & \quad \times [1 - (1 - x_2)(1 - x_3)(1 - x_4)][1 - (1 - x_4)(1 - x_5)] \\ = & (x_1 + x_2 - x_1x_2)(x_4 + x_5 - x_4x_5) \\ & \quad \times [1 - (1 - x_1)(1 - x_3)(1 - x_5)][1 - (1 - x_2)(1 - x_3)(1 - x_4)]. \end{aligned}$$

This does not look like the structure function based on the parallel representation of series systems, but it can be shown that they are algebraically identical to each other.

Example 2.1.6. **[HARDER!] (Given minimal path sets, find minimal cut sets)** You are given that the minimal path sets of a system are $\{1,5\},\{2,5\},\{1,3,4\}$, and $\{2,3,4\}$.

Calculate the number of minimal cut sets of the system.

A. 1

B. 2

C. 3

D. 4

E. 5

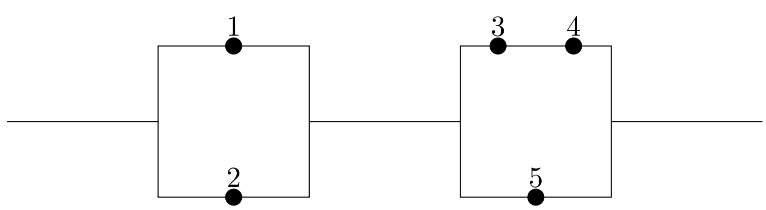
Solution 1. The key to solving this example is to realize that a cut set must have *at least one element* of every minimal path set. If this is not the case, then the functioning of all components in the excluded minimal path set will guarantee the functioning of the whole system, even if all components in a minimal cut set fail. This results in a contradiction.

Choosing one element from each of the four minimal path sets and ensuring minimality (this requires experimentation), we get three minimal cut sets:

$$\begin{array}{lllllll}
\{\boxed{1},5\}, & \{\boxed{2},5\}, & \{\boxed{1},3,4\}, & \{\boxed{2},3,4\} & \to & \{1,2\} & \\
\{1,\boxed{5}\}, & \{2,\boxed{5}\}, & \{1,\boxed{3},4\}, & \{2,\boxed{3},4\} & \to & \{3,5\} & \textbf{(Answer: C)} \\
\{1,\boxed{5}\}, & \{2,\boxed{5}\}, & \{1,3,\boxed{4}\}, & \{2,3,\boxed{4}\} & \to & \{4,5\} &
\end{array}$$

□

Solution 2 (Minimal path sets → system → minimal cut sets). With some experimentation, we can do some reverse engineering and construct the system from the given minimal path sets:



Visually inspecting the system, we find that there are three minimal cut sets, $\{1,2\},\{3,5,\},\{4,5\}$. **(Answer: C)** □

Remark. In this example, the system is simple enough so that reverse engineering used in Solution 2 is possible. For slightly more complex systems, reverse engineering will be very difficult, if not impossible (see Problems 2.4.6 and 2.4.7, which are exercises from Ross). That is why the technique illustrated in Solution 1 is still valuable.

2.2 Reliability of Systems of Independent Components

MAS-I KNOWLEDGE STATEMENT(S)

h. Bridge system and defining path to failure

i. Random graphs and defining path to failure

l. Method of inclusion and exclusion as applied to failure time estimates

OPTIONAL SYLLABUS READING(S)

Ross, Sections 9.3 and 9.4

Reliability. Probability enters our conversations in this section, where we would like to determine the probability that the overall system is functioning at a given time point of interest. Assuming that the states of the components $\mathbf{X} = (X_1, \ldots, X_n)$ (Note: We switch notation from x_i to capital letter X_i because the states are now random variables) are independent, we aim to compute the *reliability* (or reliability function) of the system defined by

$$\boxed{r(\mathbf{p}) = \mathrm{E}[\phi(\mathbf{X})] = \Pr\left(\phi(\mathbf{X}) = 1\right),}$$

which is a function of $\mathbf{p} = (p_1, \ldots, p_n)$, the vector of component reliabilities. In short, the reliability of a system is simply the probability that it will function at a reference time point.

As the following systems show, the ability to write the structure function of a system proficiently is the key to success when it comes to calculating reliability.

- *Series systems:* The reliability of a series system is

$$r(\mathbf{p}) = \Pr(X_i = 1 \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^{n} \Pr(X_i = 1) = \boxed{\prod_{i=1}^{n} p_i}, \tag{2.2.1}$$

  in which the independence of the states of the components is used in the second equality. This expression is also the same as the structure function of the series system with all x_i's replaced by p_i's upon taking expectation:

$$r(\mathbf{p}) = \mathrm{E}[X_1 X_2 \cdots X_n] \stackrel{\text{(independence)}}{=} \mathrm{E}[X_1]\mathrm{E}[X_2] \cdots \mathrm{E}[X_n] = \prod_{i=1}^{n} p_i.$$

- *Parallel systems:* The reliability of a parallel system is

$$r(\mathbf{p}) = \Pr(X_i = 1 \text{ for some } i) = 1 - \Pr(X_i = 0 \text{ for all } i) = \boxed{1 - \prod_{i=1}^{n}(1 - p_i)}. \tag{2.2.2}$$

When $n = 2$, the preceding expression simplifies to[iii]

$$\boxed{p_1 + p_2 - p_1 p_2,} \tag{2.2.3}$$

which is a simple result you may consider remembering.

Example 2.2.1. (CAS Exam S Fall 2016 Question 8: Reliability of a parallel system with uniform components) For a parallel system with two independent machines, you are given the following information:

- The hazard rate for each machine is:

$$\mu_x = \frac{1}{100 - x}, \quad \text{for } 0 \le x < 100;\ x \text{ in months}$$

- One machine has worked for 40 months and the other machine has worked for 60 months.

Calculate the probability that the system will function for 20 more months.

A. Less than 0.35

B. At least 0.35, but less than 0.55

C. At least 0.55, but less than 0.75

D. At least 0.75, but less than 0.95

E. At least 0.95

Solution. From the given hazard rate (or failure rate), we can determine the survival function of a *newborn* machine:

$$S(x) = \exp\left(-\int_0^x \frac{1}{100-t}\,\mathrm{d}t\right) = \exp[\ln(100-t)|_0^x] = 1 - \frac{x}{100}$$

for $0 \le x \le 100$. The probability that the machine aged 40 months will function for 20 more months is

$$p_1 = \frac{S(\overbrace{60}^{40+20})}{\underbrace{S(40)}_{\substack{\text{conditional on living}\\ \text{beyond 40 months}}}} = \frac{1 - 60/100}{1 - 40/100} = \frac{2}{3}$$

and the probability that the machine aged 60 months will function for 20 more months is

$$p_2 = \frac{S(80)}{S(60)} = \frac{1 - 80/100}{1 - 60/100} = \frac{1}{2}.$$

[iii] Students with more training in set theory may associate $p_1 + p_2 - p_1 p_2$, with the aid of a Venn diagram, with the probability of the union of the set where Component 1 works and the set where Component 2 works. The subtraction by $p_1 p_2$ serves to avoid double counting.

By (2.2.3), the reliability is

$$p_1 + p_2 - p_1 p_2 = \frac{2}{3} + \frac{1}{2} - \frac{2}{3}\left(\frac{1}{2}\right) = \boxed{\frac{5}{6} = 0.8333}. \quad \textbf{(Answer: D)}$$

□

Remark. (i) In fact, the remaining lifetime of a machine aged x months is uniformly distributed over $[0, 100 - x]$.

(ii) Suppose that the last line of the question is changed to "Calculate the probability that the system will function for *less than* 20 months." With $q_1 = 1 - p_1 = 1/3$, $q_2 = 1 - p_2 = 1/2$, you may be tempted to calculate the probability as

$$q_1 + q_2 - q_1 q_2 = \frac{2}{3},$$

which is not the same as one minus the answer in the original version of the example, namely $1 - 5/6 = 1/6$. In other words, we cannot simply replace the p's by the q's to find the probability that the system will fail. In fact, in terms of the q's the probability that the system will function for less than 20 months is $q_1 q_2 = 1/6$.

Example 2.2.2. **[HARDER!] (CAS Exam MAS-I Fall 2019 Question 7: Reliability of a series system with exponential and uniform components)** You are given the following information regarding a series system with two independent machines, X and Y:

- The hazard rate function, in years, for machine i is denoted by $r_i(t)$
- $r_X(t) = \ln(1.06)$, for $x > 0$
- $r_Y(t) = \frac{1}{20-t}$, for $0 < y < 20$
- Both machines are currently three years old

Calculate the probability that the system fails when the machines are between five and nine years old.

A. Less than 0.305

B. At least 0.305, but less than 0.315

C. At least 0.315, but less than 0.325

D. At least 0.325, but less than 0.335

E. At least 0.335

Ambrose's comments: In the second and third points of this MAS-I exam question, "$x > 0$" and "$0 < y < 20$" should be "$t > 0$" and "$0 < t < 20$," respectively.

Solution. We are asked to find the reliability of the series system for five years old less the reliability for nine years old. By (1.2.3),

$$S_X(t) = \mathrm{e}^{-(\ln 1.06)t} \text{ for } t > 0, \qquad \text{and} \qquad S_Y(t) = 1 - \frac{t}{20} \text{ for } 0 < t < 20.$$

Given that the system is currently three years old, the (conditional) probability that the series system is still functioning at five years old is

$$\begin{aligned}\underbrace{\frac{S_X(5)}{S_X(3)} \times \frac{S_Y(5)}{S_Y(3)}}_{(2.2.1)} &= \mathrm{e}^{-(\ln 1.06)2} \times \frac{1-5/20}{1-3/20} \\ &= 0.785291\end{aligned}$$

and the probability that it is still functioning at nine years old is

$$\begin{aligned}\frac{S_X(9)}{S_X(3)} \times \frac{S_Y(9)}{S_Y(3)} &= \mathrm{e}^{-(\ln 1.06)6} \times \frac{1-9/20}{1-3/20} \\ &= 0.456151.\end{aligned}$$

The required probability is $0.785291 - 0.456151 = \boxed{0.3291}$. **(Answer: D)** □

Example 2.2.3. **[HARDER!] (CAS Exam MAS-I Spring 2019 Question 7: Constructing a parallel system with non-identically distributed components)** You are given the following information:

- In a toolbox there are two types of components that all perform the same function:
  - ▷ There are 4 components of type A, each with reliability of 0.600
  - ▷ There are 20 components of type B, each with reliability of 0.300
  - ▷ All components are independent
- Using only the components in this toolbox, you want to construct a parallel system with a reliability of at least 0.995

Calculate the minimum number of components needed to create this system.

A. Fewer than 3

B. At least 3, but fewer than 5

C. At least 5, but fewer than 7

D. At least 7, but fewer than 9

E. At least 9

Solution. Let n be the number of components needed.

Intuitively, it makes sense to first use components of type A, which have a higher reliability, then use components of type B if needed. If only the 4 components of type A are used to construct a parallel system, the resulting reliability, by (2.2.2), is $1-(1-0.6)^4 = 0.9774$, which is lower than 0.995 and so components of type B are also needed. For the reliability of the system to be at least 0.995, we need $1-(1-0.6)^4(1-0.3)^{n-4} > 0.995$, which gives

$$0.7^{n-4} < \frac{0.005}{0.0256} \quad \Rightarrow \quad n > 4 + \frac{\ln(0.005/0.0256)}{\ln 0.7} = 8.5788.$$

The least integral value of n is $\boxed{9}$. **(Answer: E)** □

Remark. (i) Intuition suggests and (2.2.1) and (2.2.2) both confirm that adding components to a parallel system increases its reliability, whereas adding components to a series system decreases its reliability.

(ii) If you overlook the fact that there are only 4 components of type A and mistakenly solve the inequality $1-(1-0.6)^n > 0.995$, you would get $n > 5.7824$, leading to Answer C.

- *k-out-of-n systems:* The reliability function of a general k-out-of-n system is the probability that at least k out of the n components work. In the case of i.i.d. components with $p := p_1 = \cdots = p_n$,[iv] this is

$$r(\mathbf{p}) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

For non-i.i.d. components, it is not easy to display the reliability explicitly, except when k and n are small. For $k = 2$ and $n = 3$ (i.e., a 2-out-of-3 system), the reliability function is

$$\begin{aligned} r(\mathbf{p}) &= \Pr\left(X = (1,1,1), (1,1,0), (1,0,1) \text{ or } (0,1,1)\right) \\ &= \Pr(\text{exactly 2 components work}) + \Pr(\text{exactly 3 components work}) \\ &= p_1p_2(1-p_3) + p_1p_3(1-p_2) + p_2p_3(1-p_1) + p_1p_2p_3 \\ &= \boxed{p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3}, \end{aligned} \tag{2.2.4}$$

which simplifies, in the case of i.i.d. components, to

$$r(\mathbf{p}) \overset{\text{i.i.d.}}{=} \boxed{3p^2 - 2p^3}. \tag{2.2.5}$$

Question: What is the reliability function of a general 3-out-of-4 system? (this is Example 9.14 of Ross)

Answer: $p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 - 3p_1p_2p_3p_4 \overset{\text{i.i.d.}}{=} 4p^3 - 3p^4$.

[iv] Throughout this study manual, the symbol ":=" means "is defined as."

Example 2.2.4. (CAS Exam MAS-I Spring 2018 Question 8: 3-out-of-5 system) You are given a system of five independent components, with each component having reliability of 0.90. Three-out-of-five of the components are required to function for the system to function.

Calculate the reliability of this three-out-of-five system.

A. Less than 0.96

B. At least 0.96, but less than 0.97

C. At least 0.97, but less than 0.98

D. At least 0.98, but less than 0.99

E. At least 0.99

Solution. The reliability of a 3-out-of-5 system in the i.id. case is

$$\begin{aligned} r(p) &= \sum_{i=3}^{5} \binom{5}{i} p^i (1-p)^{5-i} \\ &= 10p^3(1-p)^2 + 5p^4(1-p) + p^5, \end{aligned}$$

which, when $p = 0.9$, equals $\boxed{0.99144}$. **(Answer: E)** □

Let's work out some examples involving more sophisticated systems.

Example 2.2.5. [HARDER!] (CAS Exam S Fall 2016 Question 9: Given minimal path sets, find the reliability) You are given the following information:

- A system has two minimal path sets: $\{1, 2, 4\}$ and $\{1, 3, 4\}$.
- Reliability for components 1 and 2 is uniformly distributed from 0 to 1.
- Reliability for components 3 and 4 is uniformly distributed from 0 to 2.
- All components in the system are independent.
- You are starting at time 0.

Calculate the probability that the lifetime of the system will be less than 0.25.

A. Less than 0.30

B. At least 0.30, but less than 0.35

C. At least 0.35, but less than 0.40

D. At least 0.40, but less than 0.45

E. At least 0.45

Ambrose's comments: This past exam problem challenges you by not directly telling you the design of the system. The knowledge of minimal path sets is required.

Solution. Using the parallel representation of series systems in Section 2.1, we can express the structure function of the given system algebraically as

$$\begin{aligned}\phi(\mathbf{x}) &= \max(x_1x_2x_4, x_1x_3x_4)\\ &\overset{(2.1.1)}{=} x_1x_2x_4 + x_1x_3x_4 - (x_1x_2x_4)(x_1x_3x_4)\\ &\overset{(x_1^2=x_1, x_4^2=x_4)}{=} x_1x_2x_4 + x_1x_3x_4 - x_1x_2x_3x_4\\ &= x_1(x_2 + x_3 - x_2x_3)x_4.\end{aligned}$$

Alternatively, one may observe from the two minimal path sets that the system functions if and only if both components 1 and 4 function (series), and at least one of components 2 and 3 function (parallel). This observation also leads to the above structure function.

Now replacing x_1 and x_2 by $p_1 = p_2 = 1 - 0.25/1 = 0.75$ and x_3 and x_4 by $p_3 = p_4 = 1-0.25/2 = 0.875$, we can calculate the probability (i.e., reliability) that the system function for more than 0.25 units as

$$p_1(p_2 + p_3 - p_2p_3)p_4 = 0.75(0.75 + 0.875 - 0.75 \times 0.875)(0.875) = 0.6357.$$

Finally, the probability that the lifetime of the system will be less than 0.25 is $1 - 0.6357 = \boxed{0.3643}$. **(Answer: C)** □

Remark. (i) The four-component system is taken from Example 9.16 on page 567 of Ross. In fact, it is the same in structure as the four-component system in Figure 2.1.3 – just move component 2 therein to the right and relabel the four components.

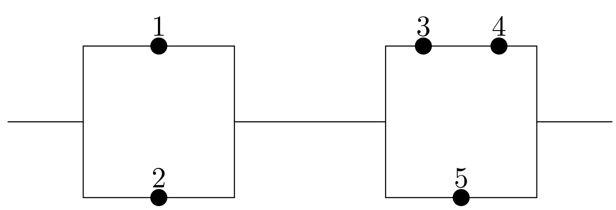
(ii) The minimal cut sets of the system are $\{1\}, \{2, 3\}$ and $\{4\}$. Giving you these three minimal cut sets immediately leads to $\phi(\mathbf{x}) = x_1(x_2 + x_3 - x_2x_3)x_4$ and may make the question too simple.

(iii) If you calculate the required probability (incorrectly) as

$$0.25(0.25 + 0.125 - 0.25 \times 0.125)(0.125) = 0.0107,$$

make sure that you read Remark (ii) of Example 2.2.1.

Example 2.2.6. **(Reliability calculations for a hybrid system)** Consider the following five-component system:



Each component functions independently with probability 0.8.

Calculate the reliability of the system.

A. Less than 0.6

B. At least 0.6, but less than 0.7

C. At least 0.7, but less than 0.8

D. At least 0.8, but less than 0.9

E. At least 0.9

Solution. This is a hybrid system consisting of a series structure and two "big" components:

1. A parallel structure with two components: Component 1 and component 2
2. A parallel structure with two components: The series structure of $\{3, 4\}$ and component 5

The structure function of the system is

$$\phi(\mathbf{x}) = \underbrace{\max(x_1, x_2)}_{\text{parallel}} \overbrace{\times}^{\text{series}} \underbrace{\max(x_3x_4, x_5)}_{\text{parallel}} \overset{(2.1.1)}{=} (x_1 + x_2 - x_1x_2)(x_3x_4 + x_5 - x_3x_4x_5).$$

Replacing the x_i's by the Bernoulli random variables X_i's and taking expectations, we have

$$r(p) = (2p - p^2)(p^2 + p - p^3) \overset{(p=0.8)}{=} \boxed{0.89088}. \quad \textbf{(Answer: D)}$$

□

Remark. (If you are interested...) You may wonder: Can this example be done by identifying the structure function of the hybrid system using the parallel representations of series systems, (2.1.2), and taking expectation of the random states X_i's, as in the preceding example? The answer is affirmative, but the solution will become much more tedious.

The minimal paths of the hybrid system are $\{1, 5\}, \{2, 5\}, \{1, 3, 4\}, \{2, 3, 4\}$. By (2.1.2), the structure function is

$$\begin{aligned} \phi(\mathbf{x}) &= \max(x_1x_5, x_2x_5, x_1x_3x_4, x_2x_3x_4) \\ &\overset{(2.1.1)}{=} 1 - (1 - x_1x_5)(1 - x_2x_5)(1 - x_1x_3x_4)(1 - x_2x_3x_4). \end{aligned}$$

You may be tempted to take expectation and convert all of the x_i's in this expression to $p = 0.8$. This is not correct because although the X_i's are mutually independent, the four terms in the product $(1 - X_1X_5)(1 - X_2X_5)(1 - X_1X_3X_4)(1 - X_2X_3X_4)$ involve duplicated X_i's. For example, X_5 appears in both the first and second terms. Without independence, the expectation of a product is no longer the product of expectations, i.e., we generally have

$$\mathrm{E}[(1 - X_1X_5)(1 - X_2X_5)(1 - X_1X_3X_4)(1 - X_2X_3X_4)]$$

$$\begin{aligned}
&\neq \mathrm{E}[(1-X_1X_5)]\mathrm{E}[(1-X_2X_5)]\mathrm{E}[(1-X_1X_3X_4)]\mathrm{E}[(1-X_2X_3X_4)] \\
&= (1-p^2)^2(1-p^3)^2.
\end{aligned}$$

For a correct solution, we have no choice but to expand the product and simplify the structure function, quite laboriously, as

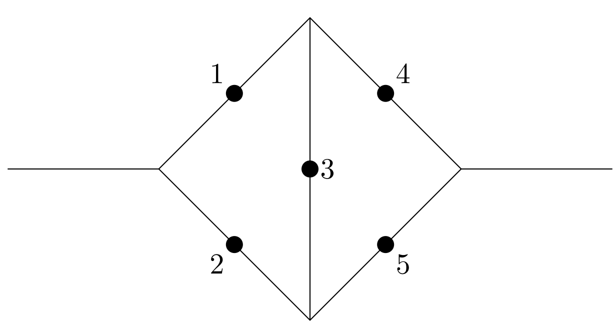
$$\begin{aligned}
\phi(\mathbf{x}) &= 1-(1-x_1x_5)(1-x_2x_5)(1-x_1x_3x_4)(1-x_2x_3x_4) \\
&\overset{(x_i^2=1)}{=} 1-(1-x_1x_5-x_2x_5+x_1x_2x_5)(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4) \\
&= 1-[(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4) \\
&\qquad +(-x_1x_5+x_1x_3x_4x_5+\underbrace{x_1x_2x_3x_4x_5-x_1x_2x_3x_4x_5}_{\text{cancel}}) \\
&\qquad +(-x_2x_5+x_1x_2x_3x_4x_5+x_2x_3x_4x_5-x_1x_2x_3x_4x_5) \\
&\qquad +(x_1x_2x_5-x_1x_2x_3x_4x_5\underbrace{-x_1x_2x_3x_4x_5+x_1x_2x_3x_4x_5}_{\text{cancel}})] \\
&= 1-[(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4)+(-x_1x_5+x_1x_3x_4x_5) \\
&\qquad +(-x_2x_5+x_2x_3x_4x_5)+(x_1x_2x_5-x_1x_2x_3x_4x_5)].
\end{aligned}$$

Note that each term in the preceding expression involves the x_i's of distinct components. Replacing the x_i's by the random variables X_i's and taking expectation, we have

$$\begin{aligned}
r(p) &= 1-[(1-2p^3+p^4)+(-p^2+p^4)+(-p^2+p^4)+(p^3-p^5)] \\
&= 2p^2+p^3-3p^4+p^5 \\
&\overset{(p=0.8)}{=} \boxed{0.89088}. \quad \textbf{(Answer: D)}
\end{aligned}$$

The message here is that the use of (2.1.2) or (2.1.3) to construct the structure function of a system is correct, but often not the most efficient method, especially when the number of minimal path or cut sets is larger than three. Visual inspection is often what you will use on an exam!

Example 2.2.7. [HARDER!] (Based on Exercise 9.14 of Ross: Reliability of a bridge system) Consider the following bridge system in which all components function independently with probability 0.7.



Calculate the reliability of the bridge system.

A. Less than 0.60

B. At least 0.60, but less than 0.70

C. At least 0.70, but less than 0.80

D. At least 0.80, but less than 0.90

E. At least 0.90

Solution. Before calculating the numerical answer for this question, for edification we pursue higher generality and assume that component i functions with probability p_i, for $i = 1, 2, \ldots, 5$. The trick to evaluate the reliability of a bridge system easily is to condition on whether component 3 (the "bridge") is working.

Case 1. If component 3 is working, then the whole system functions if and only if both of (X_1, X_2) and (X_4, X_5), each of which is considered as a parallel system *per se*, work. Such a probability is

$$\begin{aligned} & \Pr(\max(X_1, X_2) = \max(X_4, X_5) = 1) \\ = \; & \Pr(\max(X_1, X_2) = 1) \Pr(\max(X_4, X_5) = 1) \\ = \; & (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5). \end{aligned}$$

Case 2. If component 3 is not working, then the whole system functions if and only if at least one of (X_1, X_4) and (X_2, X_5), each of which is considered as a series system, works. Such a probability is

$$\Pr(\max(X_1 X_4, X_2 X_5) = 1) = p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5.$$

By the law of total probability, the unconditional probability that the bridge system functions is

$$\begin{aligned} & p_3 \Pr(\max(X_1, X_2) = \max(X_4, X_5) = 1) + (1 - p_3) \Pr(\max(X_1 X_4, X_2 X_5) = 1) \\ = \; & p_3 (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5). \end{aligned}$$

When $p_i = p$ for all $i = 1, \ldots, 5$, then the reliability of the bridge system reduces to

$$\begin{aligned} p(2p - p^2)^2 + (1 - p)(2p^2 - p^4) &= p(4p^2 - 4p^3 + p^4) + (2p^2 - 2p^3 - p^4 + p^5) \\ &= 2p^2 + 2p^3 - 5p^4 + 2p^5. \end{aligned}$$

At $p = 0.7$, the reliability of the bridge system is

$$2(0.7)^2 + 2(0.7)^3 - 5(0.7)^4 + 2(0.7)^5 = \boxed{0.8016}. \quad \textbf{(Answer: D)}$$

□

2.4 End-of-chapter Problems

Typical systems

Problem 2.4.1. **(Ross, Exercise 9.4: Basic practice with structure function)** *Please refer to the book for the question statements.*

Solution. (b) Note that components 1, 6 and at least one of $\{2,4\}$ and $\{3,5\}$ are necessary for the system to work. The structure function is

$$\phi(\mathbf{x}) = \boxed{x_1 \max(x_2x_4, x_3x_5)x_6}.$$

(c) The structure function is

$$\phi(\mathbf{x}) = \boxed{\max(x_1, x_2x_3)x_4}.$$

□

Remark. (i) The diagram in part (a) of the exercise is suspected to be incorrect.

(ii) These systems are so simple that the use of minimal path and cut sets to determine the structure function is not necessary.

Problem 2.4.2. **(Based on Question 8 of Fall 2017 Exam S (Example 2.1.2): k-out-of-n systems)** A 3-out-of-50 system is placed in series with a 48-out-of-50-system to form a combined system.

Calculate the number of minimal path sets for the combined system.

A. Fewer than 250,000

B. At least 250,000, but fewer than 500,000

C. At least 500,000 but fewer than 10,000,000

D. At least 10,000,000, but fewer than 25,000,000

E. At least 25,000,000

Solution. Recall from Example 2.1.2 that a parallel system has $\binom{n}{1} = n$ minimal path sets. In contrast, a series system only has 1 minimal path set, corresponding to all of the n components (recall that a series system works if and only if all components work).

Back to this example, the 3-out-of-50 system and 48-out-of-50 have $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets, respectively. When the two systems are put in series, a minimal path set of the overall series system can be constructed from taking a minimal path set from each of the two systems, then concatenating the two selected minimal path sets. The number of minimal path sets increases considerably to $19,600 \times 1,225 = \boxed{24,010,000}$. **(Answer: D)** □

Problem 2.4.3. (Ross, Exercise 9.8: Minimal path and cut sets of a hybrid system) *Please refer to the book for the question statements.*

Solution.

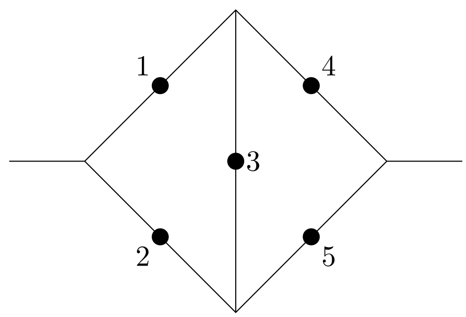
- Minimal path sets: $\boxed{\{1,3,5\},\{1,3,6\},\{2,4,5\},\{2,4,6\}}$.
- Minimal cut sets: $\boxed{\{1,2\},\{3,4\},\underbrace{\{1,4\},\{2,3\}}_{\text{Don't omit these two!}},\{5,6\}}$.

□

Remark. A by-product of this exercise is that the number of minimal path sets and the number of minimal cut sets of a system are not the same in general.

Use the following information for the next two problems.

Consider the bridge system below.



Problem 2.4.4. (Based on Example 9.8 of Ross: Identification of minimal path set for a bridge system) Determine which of the following is a minimal path set of the bridge system.

A. $\{1,2\}$

B. $\{3,5\}$

C. $\{1,2,3\}$

D. $\{2,3,4\}$

E. $\{1,2,3,4,5\}$

Solution. Only D and E guarantee that the bridge system can function when all components in D and E operate. Among D and E, only D is a minimal path set—even if components 1 and 5 are deleted from $\{1,2,3,4,5\}$, the functioning of the remaining components, namely 2, 3 and 4, is still sufficient for the functioning of the bridge system. **(Answer: D)** □

Remark. The bridge system has 4 minimal path sets: $\{1,4\},\{2,5\},\{1,3,5\}$, and $\{2,3,4\}$.