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Mathematics of Investment & Credit

8th

Eighth Edition

Samuel Broverman, Ph.D., ASA

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Mathematics of Investment & Credit

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Request for permission should be addressed to:
ACTEX Learning
PO Box 69
Greenland, NH 03840
support@actexmadriver.com

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Preface

While teaching an intermediate level university course in mathematics of investment over a number of years, I found an increasing need for a textbook that provided a thorough and modern treatment of the subject, while incorporating theory and applications. This book is an attempt (as an 8th edition, it must be an eighth attempt) to satisfy that need. It is based, to a large extent, on notes that I have developed while teaching and my use of a number of textbooks for the course. The university course for which this book was written has also been intended to help students prepare for the mathematics of investment topic that is covered on one of the professional examinations of the Society of Actuaries and the Casualty Actuarial Society. A number of the examples and exercises in this book are taken or adapted from questions on past SOA/CAS examinations.

As in many areas of mathematics, the subject of mathematics of investment has aspects that do not become outdated over time, but rather become the foundation upon which new developments are based. The traditional topics of compound interest and dated cashflow valuations, and their applications, are developed in the first five chapters of the book. In addition, in Chapters 6 to 10, a number of topics are introduced which have become of increasing importance in modern financial practice over the past number of years. The past four decades or so have seen a great increase in the use of derivative securities, particularly financial options. The subjects covered in Chapters 6 to 9, such as the term structure of interest rates, interest rate swaps and forward contracts, form the foundation for the mathematical models used to describe and value derivative securities, which are introduced in Chapter 10. This 8th edition expands on and updates the 7th edition's coverage of options, particularly the binomial model for valuing options.

The purpose of the methods developed in this book is to facilitate financial valuations. This book emphasizes a direct calculation approach, assuming that the reader has access to a financial calculator with standard financial functions as well as more sophisticated computer facilitated financial calculations. Computer routines can do such calculations much faster and more efficiently than direct calculation, but it is my strong opinion that anyone using automated calculation to determine crucial financial information must fully understand the mathematical theory underlying that calculation, The mathematical background required for the book is a course in calculus at the freshman level. Chapters 8 and 10 cover a couple of topics that involve the notion of probability, but mostly at an elementary level. A very basic understanding of probability concepts should be sufficient background for those topics.

The topics in the first five Chapters of this book are arranged in an order that is similar to traditional approaches to the subject, with Chapter 1 introducing the various measures of interest rates, Chapter 2 developing methods for valuing a series of payments, Chapter 3 considering amortization of loans, Chapter 4 covering bond valuation, and Chapter 5 introducing the various methods of measuring the rate of return earned by an investment.

The content of this book is probably more than can reasonably be covered in a one-semester course at an introductory or even intermediate level, but it might be possible for the material found in the SOA/CAS FM (Financial Mathematics) Exam to be covered in a one semester course. At the University of Toronto, the contents of this books are covered in two consecutive one-semester courses at the Sophomore (2nd year) level.

I would like to acknowledge the support of the Actuarial Education and Research Foundation, which provided support for the early stages of development of this book. I would also like to thank those who provided so much help and insight in the earlier and current editions of this book: John Mereu, Michael Gabon, Steve Linney, Walter Lowrie, Srinivasa Ramanujam, Peter Ryall, David Promislow, Robert Marcus, Sandi Lynn Scherer, Marlene Lundbeck, Richard London, David Scollnick. Robert Alps, Sam Cox and Yijia Liu.

I would like to acknowledge ACTEX Learning for their great support for this book over the years and particularly for their editorial and technical support.

Finally, I am grateful to have had the continuous support of my wife, Sue Foster, throughout the development of each edition of this book.

Samuel A. Broverman, ASA, Ph.D. University of Toronto, May 2023

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VALUATION OF ANNUITIES

"A nickel ain't worth a dime anymore."

— Yogi Berra, former NY Yankee catcher and Baseball Hall of Fame member

2.0 Introduction

Many financial transactions involve a series of payments, such as periodic dividend payments to someone owning common stock, monthly payments on a loan, or annual interest payments on a coupon bond. It is often the case (as in the loan and bond examples) that the payments are made at regularly scheduled intervals of time. In the examples in Chapter 1 that dealt with transactions involving more than one payment, each payment was treated and valued separately. When a transaction involves a number of payments made in a systematic way, it is often possible to apply algebraic methods to simplify the valuation of the series. In this chapter we will develop methods for valuing a series of payments.

Prior to the availability of sophisticated calculators and computer spreadsheet programs, it was important to have algebraic representations for series of payments that required as little calculation by hand as possible. Many of the methods developed in the past are no longer important for calculation purposes, but some remain useful for the insight that they may provide in analyzing and valuing a series of payments.

The generic term used to describe a series of periodic payments is **annuity**. In a life insurance context, an annuity is a "life-contingent" series of payments that are contingent on the survival of a specific individual or group of individuals. The more precise term for a series of payments that are not contingent on the occurrence of any specified events is **annuity-certain** (an annuity whose payments will definitely be made). Since this book deals almost entirely with annuities-certain, we shall use the term annuity to refer to an annuity-certain, unless otherwise specified.

The calculations in many of the examples presented here can be done in an efficient way using a financial calculator or computer spreadsheet program. The presentation here emphasizes understanding the underlying principles and algebraic relationships involved in annuity valuation.

A key algebraic relationship used in valuing a series of payments is the geometric series summation formula

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x} = \frac{x^{k+1} - 1}{x - 1}. \tag{2.1}$$

This is illustrated in the following example.

EXAMPLE 2.1 *(Accumulation of a level payment annuity)*

The federal government sends Smith a family allowance payment of \$30 every month for Smith's child. Smith deposits the payments in a bank account on the last day of each month. The account earns interest at the annual rate of 9% compounded monthly and the interest is paid into the account on the last day of each month. If the first payment is deposited on May 31, 2012, what is the account balance on December 31, 2023, including the payment just made and interest paid that day?

SOLUTION

The following line diagram illustrates the accumulation in the account from one deposit to the next.

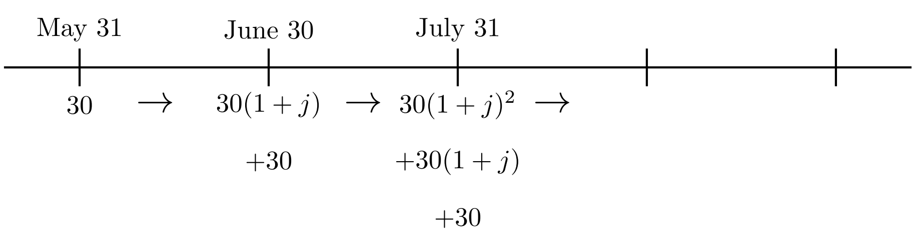


Figure 2.1

The one-month compound interest rate is $j = .0075$. The balance in the account on June 30, 2012, including the payment just deposited and the accumulated value of the May 31 deposit is

$$C_2 = 30(1 + j) + 30 = 30\left[(1 + j) + 1\right].$$

The balance on July 31, 2012 is

$$C_3 = C_2(1 + j) + 30 = 30\left[(1 + j) + 1\right](1 + j) + 30 = 30\left[(1 + j)^2 + (1 + j) + 1\right].$$

Continuing in this way we see that the balance just after the m^{th} deposit is $C_m = 30\left[(1+j)^{m-1} + \cdots + (1+j)^2 + (1+j) + 1\right]$, which is the accumulation of those first m deposits. By applying the geometric series formula, the balance on December 31, 2023, just after the 140^{th} deposit is

$$\begin{aligned} &30\left[(1+j)^{139} + (1+j)^{138} + \cdots + (1+j) + 1\right] \\ &\qquad = 30\left[\frac{(1+j)^{140} - 1}{(1+j) - 1}\right] \\ &\qquad = 30\left[\frac{(1.0075)^{140} - 1}{.0075}\right] = 7385.91. \end{aligned}$$

□

2.1 Level Payment Annuities

2.1.1 Accumulated Value of an Annuity

In Example 2.1, the expression for the aggregate accumulated value on December 31, 2023 is

$$30(1+j)^{139} + 30(1+j)^{138} + \cdots + 30(1+j) + 30.$$

This is the sum of the accumulated values of the individual deposits. $30(1+j)^{139}$ is the accumulated value on December 31, 2023 of the deposit made on May 31, 2012, $30(1+j)^{138}$ is is the accumulated value on December 31, 2023 of the deposit made on June 30, 2012, and so on.

Let us consider a series of n payments (or deposits) of amount 1 each, made at equally spaced intervals of time, and for which interest is at compound rate i per payment period, with interest credited on payment dates. The accumulated value of the series of payments, valued at the time of (and including) the final payment, can be represented as the sum of the accumulated values of the individual payments. This is

$$(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 = \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}. \quad (2.2)$$

This is illustrated in the following diagram. We can see from the diagram that since the valuation point is the time that the n^{th} deposit is made, this is actually $n-1$ periods after the first deposit. Therefore the first deposit has grown with compound interest for $n-1$ periods.

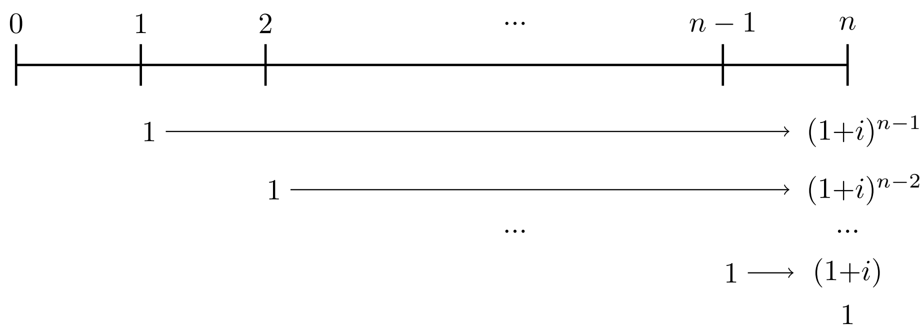


Figure 2.2

There is standard actuarial notation and terminology that describes this annuity.

Definition 2.1 – Accumulated Value of an n-Payment Annuity of 1 Per Period

The symbol $s_{\overline{n}|i}$ denotes the accumulated value, at the time of (and including) the final payment of a series of n payments of 1 each made at equally spaced intervals of time, where the rate of interest per payment period is i.

$$\begin{aligned} s_{\overline{n}|i} &= (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 \\ &= \sum_{t=0}^{n-1} (1+i)^t = \frac{(1+i)^n - 1}{i} \end{aligned} \tag{2.3}$$

The symbol $s_{\overline{n}|i}$ and other related annuity symbols provide notation that can be used to efficiently represent transactions that involve a series of level payments. If there is no possibility of confusion with other interest rates in a particular situation, the subscript i is omitted from $s_{\overline{n}|i}$ and the accumulated value is denoted $s_{\overline{n}|}$, without the subscript i. The number of payments in the series is called the **term of the annuity**, and the time between successive payments is called the **payment period**, or **frequency**. Note that for any interest rate i, $s_{\overline{1}|i} = 1$, but if $i > 0$ and $n > 1$, then $s_{\overline{n}|i} > n$ because of interest accumulation on earlier deposits.

It should be emphasized that the $s_{\overline{n}|i}$ notation can be used to express the accumulated value of an annuity provided the following conditions are met:

(1) the interest rate is constant at i per payment period for the full term of the annuity

(2) the payments are of equal amount;

(3) the payments are made at equal intervals of time, with the same frequency as the interest rate is compounded;

(4) the accumulated value is found at the time of and including the final payment.

This series of payments is referred to in actuarial terminology as an accumulated **annuity-immediate**. When we are finding the accumulated value of a series of payments, the phrase "annuity-immediate" indicates that the value is being calculated at the time of the final payment. We often see a series of payments described with the phrase "payments occur at the end of each year (or month, etc.)," with a valuation made at the end of n years, which makes the valuation point the time of the final payment. The conventional interpretation of this phrase is to regard the valuation as an accumulated annuity-immediate.

Figure 2.3 plots the accumulated value of the annuity as a function of the number of annuity payments (using a 10% interest rate).

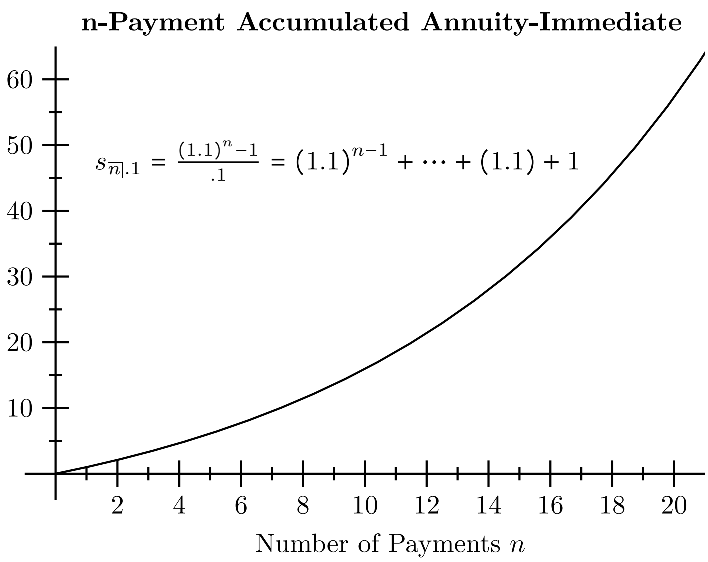


Figure 2.3

General reasoning suggests that if the interest rate is increased, accumulated values increase. Figure 2.4 plots the accumulated value of a 20-payment annuity as a function of the interest rate. If the interest rate is 0, then there is no interest accumulation, and the accumulated value of the annuity is

$$s_{\overline{20}|i=0} = (1+0)^{19} + (1+0)^{18} + \cdots + (1+0) + 1 = 20.$$

This accumulated annuity value is just the sum of the 20 annuity payments of amount 1 each.

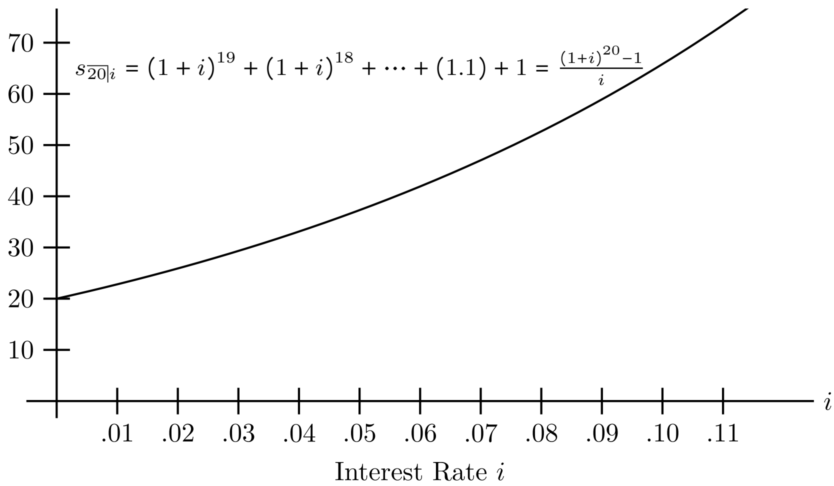


Figure 2.4

Equation 2.3 can be rewritten as

$$(1+i)^n = i \times s_{\overline{n}|i} + 1 = i \times \left[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1\right] + 1.$$

We can interpret this expression in the following way. Suppose that a single amount of 1 is invested at time 0 at periodic interest rate i, so that an interest payment of i is generated at the end of each period. Suppose further that each interest payment is reinvested and continues to earn interest at rate i. This is allowed to continue for n periods. Then the accumulation of the reinvested interest, along with the original amount 1 invested (the right hand side of the equation above), must be equal to the compound accumulation of 1 at rate i per period invested for n periods.

EXAMPLE 2.2 (*Accumulated value of an annuity-immediate*)

What level amount must be deposited on May 1 and November 1 each year from 2018 to 2025, inclusive, to accumulate to 7000 on November 1, 2025 if the nominal annual rate of interest compounded semi-annually is 9%?

SOLUTION

A first step in translating the verbal description of this annuity into an algebraic form is to determine the number of deposits being made. There are a total of 16 deposits (2 per year for each of the 8 years from 2018 to 2025 inclusive) and they occur every $\frac{1}{2}$ year. As a second step, we note that the $\frac{1}{2}$-year interest rate is 4.5%, and the $\frac{1}{2}$-year payment period corresponds to the $\frac{1}{2}$-year interest compounding period. If the level amount deposited every $\frac{1}{2}$-year is denoted by

X, the accumulated value of the deposits at the time of the 16^{th} (final) deposit is

$$X \times \left[(1.045)^{15} + (1.045)^{14} + \cdots + 1.045 + 1\right]$$
$$= X \times \frac{(1.045)^{16} - 1}{.045} = X\, s_{\overline{16}|.045} = 22.719337X$$

(note that the factor $(1.045)^{15}$ arises as a result of there being 15 half-year periods from the time of the first deposit on May 1, 2018 to the time of the 16^{th} deposit on November 1, 2025). Then

$$X = \frac{7000}{s_{\overline{16}|.045}} = \frac{7000 \times .045}{(1.045)^{16} - 1} = \frac{7000}{22.719337} = 308.11. \qquad \square$$

All financial calculators have functions that calculate the accumulated value of an annuity at the time of the final payment if the payment amount, number of payments, and interest rate are known.

2.1.1.1 Accumulated Value of an Annuity Some Time after the Final Payment

After a series of deposits to an account is completed, the balance in the account can continue to accumulate with interest only. The following example illustrates this.

EXAMPLE 2.3 (*Accumulated value of an annuity some time after the final payment*)

Suppose that in Example 2.1, Smith's child is born in April, 2012 and the first payment is received in May and deposited at the end of May. The payments continue and the deposits are made at the end of each month until (and including the month of) the child's 16^{th} birthday. The payments cease after the 16^{th} birthday, but the balance in the account continues to accumulate with interest until the end of the month of the child's 21^{st} birthday. What is the balance in the account at that time?

SOLUTION

At the end of the month of the child's 16^{th} birthday, Smith makes the 192^{nd} deposit into the account. This is at the end of April, 2028 (there are 12 deposits per year for 16 years for a total of 192 deposits). The accumulated value at that time is

$$30 \times \left[(1.0075)^{191} + (1.0075)^{190} + \cdots + (1.0075) + 1\right]$$
$$= 30 \times s_{\overline{192}|.0075} = 30 \times \frac{(1.0075)^{192} - 1}{.0075} = 12{,}792.31.$$

Five years (60 months) later, at the end of the month of the child's 21st birthday (April 30, 2033), the account will have grown, with interest only, to

$$12{,}792.31(1.0075)^{60} = 20{,}028.68. \qquad \square$$

We have seen that the value at the time of the n^{th} deposit of a series of n deposits of amount 1 each is $\frac{(1+i)^n-1}{i} = s_{\overline{n}|}$. If there are no further deposits, but the balance continues to grow with compound interest, then the accumulated value k periods after the n^{th} deposit is

$$\begin{aligned}
[(1+i)^{n-1} &+ (1+i)^{n-2} + \cdots (1+i) + 1] \times (1+i)^k \\
&= (1+i)^{n+k-1} + (1+i)^{n+k-2} + \cdots + (1+i)^{k+1} + (1+i)^k \\
&= \frac{(1+i)^n - 1}{i} \times (1+i)^k \\
&= s_{\overline{n}|} \times (1+i)^k \\
&= (\text{Value at time n}) \times (\text{growth factor from time } n \text{ to time } n+k).
\end{aligned}$$

This can be also be represented in the following way.

$$\begin{aligned}
s_{\overline{n}|} \times (1+i)^k &= \frac{(1+i)^n - 1}{i} \times (1+i)^k = \frac{(1+i)^{n+k} - (1+i)^k}{i} \\
&= \frac{(1+i)^{n-k} - 1}{i} - \frac{(1+i)^k - 1}{i} = s_{\overline{n+k}|} - s_{\overline{k}|}.
\end{aligned} \tag{2.4}$$

Using Equation 2.4 with $n = 192$ and $k = 60$, the accumulated value of the account on April 30, 2033 in Example 2.3 can be written as $30[s_{\overline{252}|} - s_{\overline{60}|}]$.

Figures 2.5a and 2.5b below illustrate the formulations given in Equation 2.4. If the annuity payments had continued to time $n+k$, which is the time of valuation, the accumulated value would be $s_{\overline{n+k}|}$. Since there are not any payments actually made for the final k payment periods, $s_{\overline{n+k}|}$ must be reduced by $s_{\overline{k}|}$, the accumulated value of k payments of 1 each ending at time $n+k$.

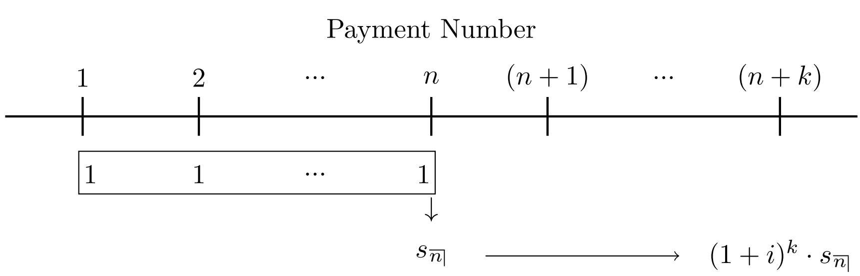


Figure 2.5a

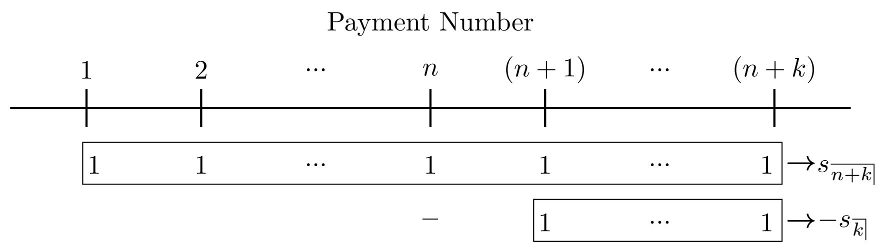


Figure 2.5b

Equation 2.4 can be reformulated as

$$s_{\overline{n+k}|} = s_{\overline{n}|}(1+i)^k + s_{\overline{k}|}. \tag{2.5}$$

Equation 2.5 shows that a series of payments can be separated into components, and the accumulated value of the entire series at a valuation point can be represented as the sum of the accumulated values (at that valuation point) of the separate component series.

2.1.1.2 Accumulated Value of an Annuity with Non-Level Interest Rates

The concept of dividing a series of payments into subgroups and valuing each subgroup separately can be applied to find the accumulated value of an annuity when the periodic interest rate changes during the term of the annuity. This is illustrated in the following modification of Example 2.1.

EXAMPLE 2.4 (*Annuity accumulation with non-level interest rates*)

Suppose that in Example 2.1 the nominal annual interest rate earned on the account changes to 7.5% (still compounded monthly) as of January 1, 2018. What is the accumulated value of the account on December 31, 2023?

SOLUTION

In a situation in which the interest rate is at one level for a period of time and changes to another level for a subsequent period of time, in order to simplify the algebraic analysis, it is necessary to separate the full term into separate time intervals over which the interest rate is constant. We first calculate the accumulated value in the account on December 31, 2017, since the nominal interest rate is level at 9% up until this point. This accumulated value is

$$30 \times \frac{(1.0075)^{68} - 1}{.0075} = 30 \times s_{\overline{68}|.0075} = 2{,}648.50.$$

From January 1, 2018 onward, the accumulation in the account can be separated into two components: the accumulation of the 2,648.50 that was on balance as of January 1, 2018, and the accumulation of the continuing deposits from January 31, 2018 onward. The monthly rate of interest is $\frac{0.075}{12} = 0.00625$ from January 1, 2018 onward, so the 2,648.50 accumulates to

$$2{,}648.50 \times (1.00625)^{72} = 4{,}147.86$$

as of December 31, 2023, and the remaining deposits continuing from January 31, 2018 accumulate to

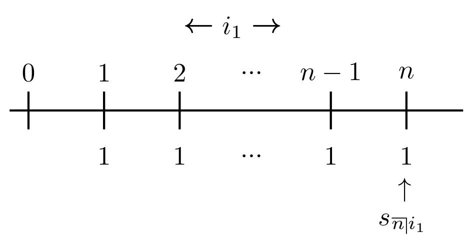
$$30 \times \frac{(1.00625)^{72} - 1}{0.00625} = 30 \times s_{\overline{72}|.00625} = 2{,}717.36,$$

for a total of $4{,}147.86 + 2{,}717.36 = 6{,}865.22$. The accumulated value on December 31, 2023 can be written as

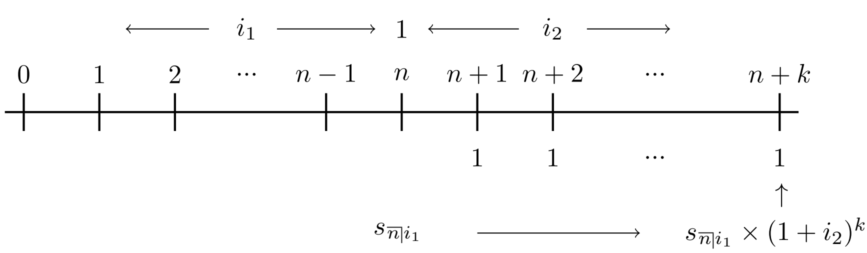
$$30 \times \left[s_{\overline{68}|.0075} \times (1.00625)^{72} + s_{\overline{72}|.00625} \right]. \qquad \square$$

We can generalize the concept presented in Example 2.4. Suppose that we consider an $n+k$-payment annuity with equally spaced payments of 1 per period and with an interest rate of i_1 per payment period up to the time of the n^{th} payment, followed by an interest rate of i_2 per payment period from the time of the n^{th} payment onward. The accumulated value of the annuity at the time of the final payment can be found in the following way.

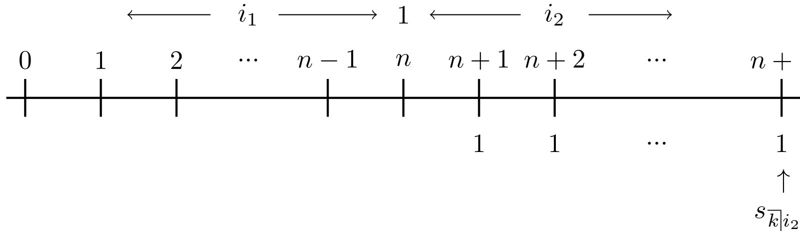
(a) The accumulated value of the first n payments valued at the time of the n^{th} is $s_{\overline{n}|i_1}$.



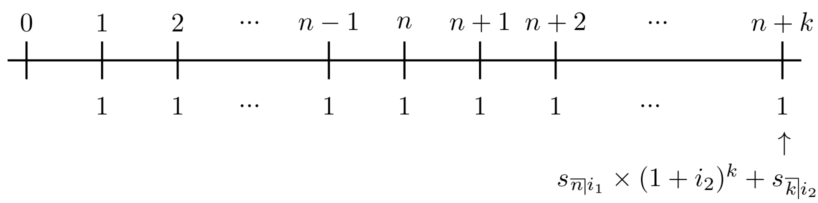
(b) The accumulated value found in part (a) grows with compound interest for an additional k periods at compound periodic interest rate i_2, to a value of $s_{\overline{n}|i} \times (1+i_2)^k$ as of time $n+k$.



(c) The accumulated value of the final k payments is $s_{\overline{k}|i_2}$.



(d) The total accumulated value at time $n+k$ is the sum of (b) and (c), and equals $s_{\overline{n}|i} \times (1+i_2)^k + s_{\overline{k}|i_2}$.



Note that if the interest rate is level over the $n+k$ periods, so that $i_2 = i_1$, then Equation 2.5 is the same as the expression in (d). This method can be extended to situations in which the interest rate changes more than once during the term of the annuity.

2.1.1.3 Accumulated Value of an Annuity With a Changing Payment

The relationship in Equation 2.5 can also be used to find the accumulated value of an annuity for which the payment amount changes during the course of the annuity. The following example illustrates this point.

EXAMPLE 2.5 *(Annuity whose payment amount changes during annuity term)*

Suppose that 10 monthly payments of 50 each are followed by 14 monthly payments of 75 each. If interest is at an effective monthly rate of 1%, what is the accumulated value of the series at the time of the final payment?

SOLUTION

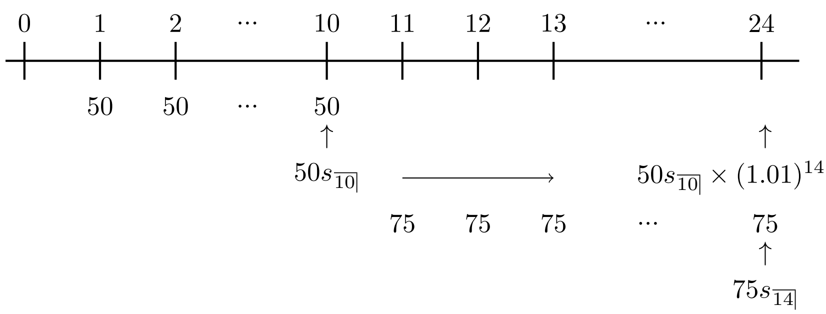
We can use the same technique as in Example 2.3 for finding the accumulated value of an annuity some time after the final payment. The accumulated value of the first 10 payments of 50 each, valued at the time of the 10$^{\text{th}}$ payment is $50s_{\overline{10}|.01}$. If we accumulate this amount for another 14 months (ignoring, for the moment the remaining 14 payments of 75 each), we get the accumulated value at time 24 (months) of the first 10 payments, which is

$$50s_{\overline{10}|.01} \times (1.01)^{14} = 601.30.$$

The value of the final 14 payments valued at the time of the 14th of those payments, which is also time 24 is

$$75s_{\overline{14}|.01} = 1{,}121.06.$$

The total accumulated value at time 24 is 1722.36. The following diagram illustrates this accumulation.



There is an alternative way of approaching the situation in Example 2.5. Note in Figure 2.6 that the original (non-level) sequence of payments can be decomposed into two separate level sequences of payments, both of which end at time 24.

Time	0	1	2	···	10	11	12	···	24
Original Series		50	50	···	50	75	75	···	75
New Series 1		50	50	···	50	50	50	···	50
New Series 2						25	25	···	25

Figure 2.6

The accumulated value (at time 24) of the alternate form of the series is $50s_{\overline{24}|.01} + 25s_{\overline{14}|.01} = 1{,}348.67 + 373.69 = 1{,}722.36.$ □

It should be emphasized again that $s_{\overline{n}|i}$ is very specific in its meaning. It is the valuation **at the time of the n-th payment of n equally spaced payments of 1 each with effective interest rate i per payment period**.

2.1.2 Present Value of an Annuity

The discussion up to now in this chapter has been concerned with formulating and calculating the accumulated value of a series of payments. We now consider the **present value** of an annuity, which is a valuation of a series of payments some time before the payments begin.

EXAMPLE 2.6 (*Present value of a series of payments*)

Smith's grandchild will begin a four year college program in one year. Smith wishes to open a bank account with a single deposit today to provide some funding for the grandchild's college education. Smith would like the grandchild to be able to withdraw 1000 from the account each year for four years, with the first withdrawal taking place one year from now, and subsequent withdrawals each year after that. Smith would like the account balance to be 0 immediately after the final withdrawal is made four years from now. The account has an annual effective interest rate of 6%. Determine the amount of the deposit Smith makes today.

SOLUTION

Suppose that the amount of the initial deposit is X. If we track the account balance after each withdrawal, we see the following:
Balance after 1$^{\text{st}}$ withdrawal:

$$X(1.06) - 1000$$

Balance after 2$^{\text{nd}}$ withdrawal:

$$[X(1.06) - 1000](1.06) - 1000 = X(1.06)^2 - 1000(1.06) - 1000$$

Balance after 3$^{\text{rd}}$ withdrawal:

$$\begin{aligned} &\left[X(1.06)^2 - 1000(1.06)\right](1.06) - 1000 \\ &\qquad = X(1.06)^3 - 1000(1.06)^2 - 1000(1.06) - 1000 \end{aligned}$$

Balance after 4$^{\text{th}}$ withdrawal:

$$X(1.06)^4 - 1000(1.06)^3 - 1000(1.06)^2 - 1000(1.06) - 1000.$$

In order for the balance to be 0 just after the 4$^{\text{th}}$ withdrawal, we must have

$$X(1.06)^4 = 1000(1.06)^3 + 1000(1.06)^2 + 1000(1.06) + 1000,$$

or equivalently,

$$\begin{aligned} X &= \frac{1000}{1.06} + \frac{1000}{(1.06)^2} + \frac{1000}{(1.06)^3} + \frac{1000}{(1.06)^4} \\ &= 1000\left[v + v^2 + v^3 + v^4\right] = 3{,}465.11. \end{aligned}$$

□

Note that in Example 2.6 there is an implicit assumption that interest is credited into the account one year after the initial deposit and every year after that.

It can be seen from Example 2.6 that the deposit amount needed is the combined present value of the four withdrawals that will be made. The **present value of an annuity** of payments is the value of the payments at the time, or some time before, the payments begin.

Consider again a series of n payments of amount 1 each, made at equally spaced intervals for which interest is at effective interest rate i per payment period. The present value of the series of payments, valued **one period before** the first payment, can be represented as the sum of the present values of the individual payments:

$$\begin{aligned}&\frac{1}{(1+i)}+\frac{1}{(1+i)^2}+\cdots+\frac{1}{(1+i)^{n-1}}+\frac{1}{(1+i)^n}\\ &\quad=v+v^2+\cdots+v^{n-1}+v^n=v\times\left[1+v+\cdots+v^{n-1}\right]\\ &\quad=v\times\frac{1-v^n}{1-v}=\frac{1-v^n}{(1+i)\left(1-\frac{1}{1+i}\right)}=\frac{1-v^n}{1+i-1}=\frac{1-v^n}{i}\end{aligned} \tag{2.6}$$

Applying Equation 2.6 to Example 2.6 we see that the present value of the four payment annuity received by Smith's grandchild is:

$$1000\left[v+v^2+v^3+v^4\right]=1000\left[\frac{1-v_{.06}^4}{.06}\right]=3{,}465.11$$

It is often the case that, as in Example 2.6, a valuation of a series of payments is done one period before the first payment. There is a specific **actuarial symbol** and terminology that represents the present value of such an annuity.

Definition 2.2 – Present Value of an n-Payment Annuity- Immediate of 1 Per Period

The symbol $a_{\overline{n}|i}$ is specifically used to denote the present value of a series of equally spaced payments of amount 1 each when the valuation point is one payment period before the payments begin and the interest rate is i per payment period.

$$a_{\overline{n}|i}=v+v^2+\cdots+v^n=\sum_{t=1}^{n}v^t=\frac{1-v^n}{i} \tag{2.7}$$

This valuation is illustrated in the following line diagram.

$$\begin{array}{cccccc}
0 & & 1 & 2 & \cdots & n \\
\hline
v & \leftarrow & 1 & & & \\
v^2 & & \leftarrow & 1 & & \\
\vdots & & & & & \\
v^n & & & & \leftarrow & 1 \\
\hline
a_{\overline{n}|i} & \leftarrow & 1 & 1 & \cdots & 1
\end{array}$$

Similar to the use of the notation $s_{\overline{n}|}$, the symbol $a_{\overline{n}|}$ can be used to express the present value of an annuity provided the following conditions are met:

(1) The interest rate is constant at i per payment period for the full term of the annuity.

(2) There are n payments of equal amount.

(3) The payments are made at equal intervals of time, with the same frequency as the frequency of interest compounding.

(4) The valuation point is one payment period before the first payment is made.

A typical situation in which the present value of an annuity-immediate arises is the repayment of a loan. In financial practice, a loan being repaid with a series of payments is structured so that the original loan amount advanced to the borrower is equal to the present value of the loan payments to be made by the borrower, and it is often the case that the first loan payment is made one payment period after the loan is made. The present value is calculated using the loan interest rate.

EXAMPLE 2.7 (*Loan repayment*)

Smith has bought a new car and requires a loan of 12,000 to pay for it. The car dealer offers Smith two alternatives on the loan:

(a) monthly payments for 3 years, starting one month after purchase, with an annual interest rate of 12% compounded monthly, or

(b) monthly payments for 4 years, also starting one month after purchase, with annual interest rate 15%, compounded monthly.

Find Smith's monthly payment and the total amount paid over the course of the repayment period under each of the two options.

SOLUTION

We denote the monthly payment under option (a) by P_1 and under option (b) by P_2. "12% compounded monthly" refers to a one month interest rate of 1%, and alternative (b) refers to a one-month interest rate of 1.25%. Since payments begin one month (one payment period) after the loan, the equations of value for the two options are

(a) $12{,}000 = P_1 \times [v_{.01} + v_{.01}^2 + \cdots + v_{.01}^{36}] = P_1 \times a_{\overline{36}|.01}$ and

(b) $12{,}000 = P_2 \times a_{\overline{48}|.0125}$.

Then

$$P_1 = \frac{12{,}000}{a_{\overline{36}|.01}} = \frac{12{,}000(.01)}{1-(1.01)^{-36}} = \frac{12{,}000}{30.107505} = 398.57,$$

and

$$P_2 = \frac{12{,}000}{a_{\overline{48}|.0125}} = \frac{12{,}000}{35.931363} = 333.97.$$

The total paid under option (a) would be $36P_1 = 14{,}348.52$, and under option (b) it would be $48 \times P_2 = 16{,}030.56$. □

The following two graphs, Figures 2.7a and 2.7b, illustrate the present value of an annuity-immediate first as a function the rate of interest, and then as a function of the number of payments. The first graph is an illustration of the fact that as interest rates increase, present value decreases. The second graph shows that the present value of a payment made far in the future is small and adds little to the present value of the series.

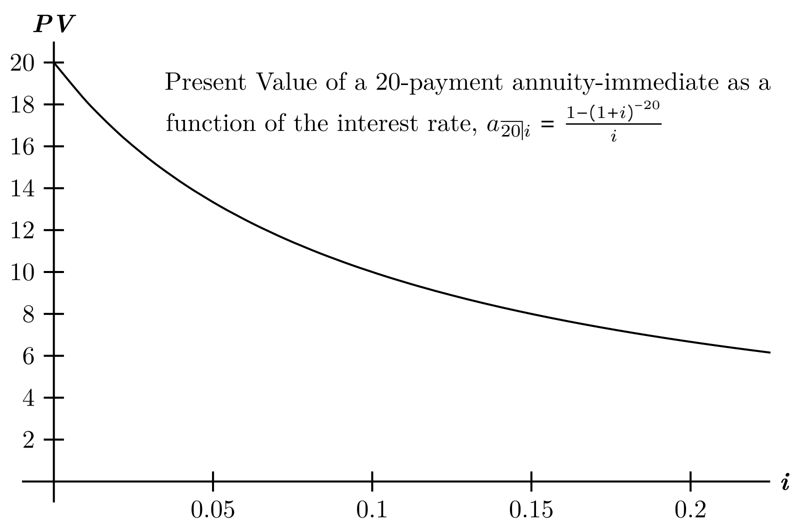


Figure 2.7a

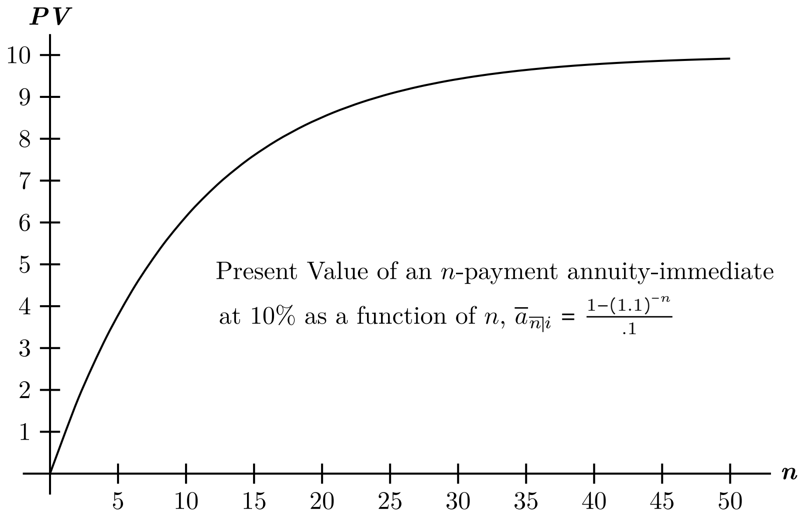


Figure 2.7b

2.1.2.1 Present Value of an Annuity Some Time Before Payments Begin

Earlier we saw how to find the value of an accumulated annuity some time after the final payment, Under other circumstances it may be necessary to find the present value of a series of payments some time before the first payment is made. This is illustrated in the following modification of Example 2.7.

EXAMPLE 2.8 (*Valuation of an annuity some time before payments begin*)

Suppose that in Example 2.7 Smith can repay the loan, still with 36 payments under option (a) or 48 payments under option (b), with the first payment made 9 months after the car is purchased in either case. Assuming interest accrues from the time of the car purchase, find the payments required under options (a) and (b).

SOLUTION

We denote the new payments under options (a) and (b) by P_1' and P_2', respectively. Then the equation of value for option (a) is

$$12{,}000 = P_1' \times \left[v^9 + v^{10} + \cdots + v^{44}\right] = P_1' \times v^8 \times a_{\overline{36}|.01},$$

which leads to

$$P_1' = \frac{12{,}000}{v^8 \times a_{\overline{36}|.01}} = (1.01)^8 \times \frac{12{,}000}{a_{\overline{36}|.01}} = (1.01)^8 \times P_1 = 431.60.$$

In a similar manner,

$$P_2' = (1.0125)^8 \times \frac{12{,}000}{a_{\overline{48}|.0125}} = (1.0125)^8 \times P_2 = 368.86.$$

Since the payments are deferred for 8 months from their original starting date in Example 2.7 (instead of starting in one month as in the original example, the payments now start in 9 months, which is 8 months later than the original start date), it should not be surprising that in each of cases (a) and (b) the new payment is equal to the old payment multiplied by the 8-month accumulation factor $(1.0125)^8$.

□

To generalize the situation in Example 2.8, suppose an n-payment annuity of 1 per period is to be valued $k+1$ payment periods before the first payment is made. The present value can be expressed as $v^{k+1} + v^{k+2} + \cdots + v^{k+n}$, which can be reformulated as

$$v^k \times \left[v + v^2 + \cdots + v^n\right] = v^k \times a_{\overline{n}|}.$$

Since $a_{\overline{n}|}$ represents the present value of the annuity one period before the first payment, the value k periods before that (for a total of $k+1$ periods before the first payment) should be $v^k \times a_{\overline{n}|}$. With a derivation similar to that for Equation 2.4, we have

$$v^k \times a_{\overline{n}|} = a_{\overline{n+k}|} - a_{\overline{k}|}. \tag{2.8}$$

Such an annuity is called a **deferred annuity**. The annuity considered in Equation 2.8 is usually called a k-period deferred, n-payment annuity-immediate of 1 per period. This present value may be denoted by ${}_{k|}a_{\overline{n}|}$. Note that for a k-period deferred annuity-immediate, the first payment comes $k+1$ periods after the valuation date, not k periods after. Equation 2.8 can be rewritten in the form

$$a_{\overline{n+k}|} = a_{\overline{k}|} + v^k \times a_{\overline{n}|} = a_{\overline{n}|} + v^n \times a_{\overline{k}|}. \tag{2.9}$$

Valuation of a deferred annuity is illustrated in the following line diagram.

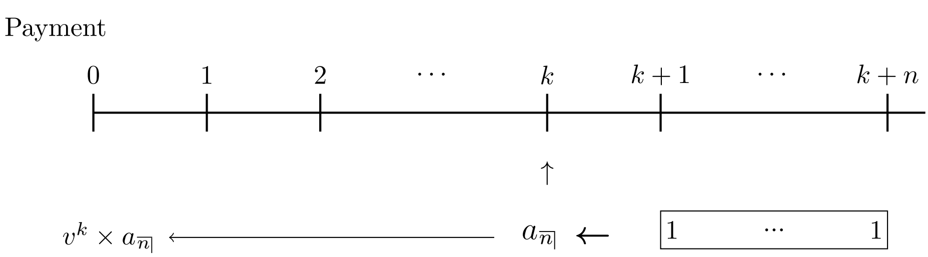


Figure 2.8

2.1.2.2 Present Value of an Annuity with Non-Level Interest Rates

Just as Equation 2.5 can be used for accumulated annuities, Equation 2.9 can be applied to find the present value of an annuity for which the interest rate changes during the term of the annuity. If we consider an $n+k$-payment annuity with equally spaced payments, with an interest rate of i_1 per period up to the time of the n^{th} payment followed by a rate of i_2 per period from the n^{th} payment onward, the present value of the annuity one period before the first payment can be found in the following manner.

(a) The present value of the first n payments valued one period before the first payment is $a_{\overline{n}|i_1}$.

(b) The present value of the final k payments valued at time n (one period before the first of the final k payments) at rate i_2 is $a_{\overline{k}|i_2}$.

(c) The value of (b) at time 0 (one period before the first payment of the entire series) at interest rate i_1 per period over the first n periods is $v_{i_1}^n \times a_{\overline{k}|i_2}$.

(d) The total present value at time 0 is the sum of (a) and (c), which is $a_{\overline{n}|i_1} + v_{i_1}^n \times a_{\overline{k}|i_2}$.

This is illustrated in the following line diagram.

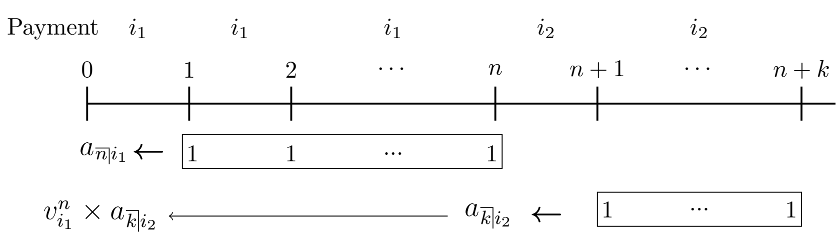


Figure 2.9

EXAMPLE 2.9 (*Annuity with non-level interest rates*)

Smith borrows \$10,000 and will repay the loan with monthly payments for three years, with the first payment to be made one month after the loan is received. The lender has the following schedule for nominal interest rates compounded monthly on the loan:

$$1^{\text{st}} \text{ year: } 6\%, \quad 2^{\text{nd}} \text{ year: } 9\%, \quad 3^{\text{rd}} \text{ year: } 12\%$$

Also, Smith's loan payments will have the following pattern:

$$1^{\text{st}} \text{ yr: } X \text{ per mo.,} \quad 2^{\text{nd}} \text{ yr: } X+25 \text{ per mo.,} \quad 3^{\text{rd}} \text{ yr: } X+50 \text{ per mo.}$$

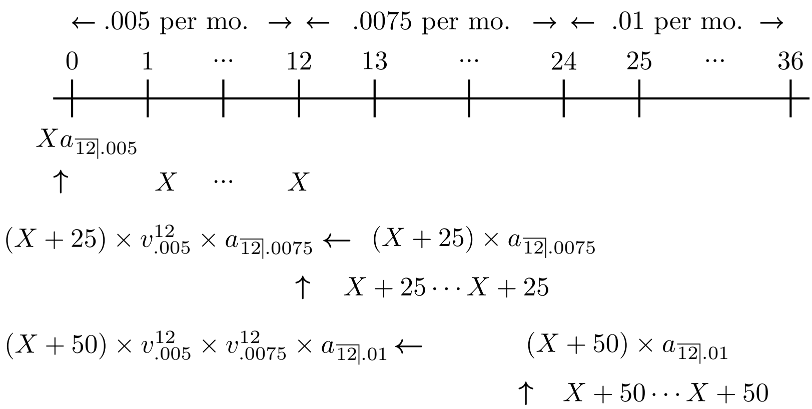
Determine X.

SOLUTION

At the time the loan is received, the present values of the first, second and third groups of 12 monthly payments are

$$Xa_{\overline{12}|.005},\ (X+25)\times v^{12}_{.005}\times a_{\overline{12}|.0075},\ \text{and}\ (X+50)\times v^{12}_{.005}\times v^{12}_{.0075}\times a_{\overline{12}|.01},$$

respectively. The total present value is set equal to the loan amount $10,000. Solving for X results in $X = 288.21$. The time diagram below illustrates the valuation of the three separate groups of 12 payments.



□

2.1.2.3 Relationship Between $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$

We now return to annuities with a level interest rate. The basic valuation point for an n-payment accumulated annuity-immediate is the time of the n^{th} payment and the accumulated value at that time is $s_{\overline{n}|i}$. The basic valuation point for the present value of an n-payment annuity-immediate is one period before the first payment, and the present value is $a_{\overline{n}|i}$. We see that the valuation point for the present value of the annuity is n periods earlier than the valuation point for the accumulated value. It follows that

$$s_{\overline{n}|i} = (1+i)^n \times a_{\overline{n}|i} \tag{2.10}$$

and

$$a_{\overline{n}|i} = v^n \times s_{\overline{n}|i}. \tag{2.11}$$

This can be easily verified algebraically by observing that

$$v^n \times s_{\overline{n}|i} = v^n \times \left[\frac{(1+i)^n - 1}{i}\right] = \frac{1-v^n}{i} = a_{\overline{n}|i}.$$